Taylor–Lagrange renormalization and gauge theories in four dimensions

2012 J. Phys. A: Math. Theor. 45 315401

(http://iopscience.iop.org/1751-8121/45/31/315401)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **45** (2012) 315401 (25pp) doi:10.1088/1751-8113/45/31/315401

Taylor–Lagrange renormalization and gauge theories in four dimensions

B Mutet[1], **P Grangé**[2] **and E Werner**[3]

[1] Laboratoire de Physique Corpusculaire, Clermont Université, BP 104448, F-63000 Clermont-Ferrand, France
[2] Laboratoire Univers et Particules, Université Montpellier II, CNRS/IN2P3, Place E Bataillon, F-34095 Montpellier Cedex 05, France
[3] Institut für Theoretische Physik, Universität Regensburg, Universitätstrasse 31, D-93053 Regensburg, Germany

E-mail: mutet@clermont.in2p3.fr, pcgrange@univ-montp2.fr and egwerner@gmx.de

Received 17 February 2012, in final form 5 June 2012
Published 23 July 2012
Online at stacks.iop.org/JPhysA/45/315401

Abstract
The treatment of gauge theories within the recently proposed Taylor–Lagrange renormalization scheme (TLRS) is examined in detail. The conservation of gauge symmetry is demonstrated directly at the physical dimension $D = 4$ for specific examples of fermion and boson self energies and vertices of QED and QCD. Comparisons with dimensional regularization and Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) substractions, improved by algebraic regularization based on the quantum action principle, exhibit clearly the important mathematical properties of the TLRS leading to conservation of this fundamental gauge symmetry.

PACS numbers: 11.10.Ef, 11.10.Gh, 11.10.−z, 03.70+k

1. Introduction

The Taylor–Lagrange renormalization scheme (TLRS) method introduced in [1, 2] is based on general considerations for fields as operator-valued distributions (OPVD). It is quite general and applies in principle to any formulation of quantum field theory (QFT). In this approach, each fermion or boson field operator is a linear functional of an appropriate test function, which is C^∞ and of the type of a partition of unity (PU). On general grounds, its mathematical properties offer the possibility of avoiding symmetry violation problems usually encountered with conventional ultraviolet (UV) regularizations. The TLRS formalism has recently been tested in the case of relativistic bound state systems in the context of covariant light-front dynamics (CLFD) [3, 4]. In this case, the symmetry in question is the independence of physical results with respect to the orientation of the lightcone (LC). In the past, this independence was also achieved with the Pauli–Villars (PV) regularization, but calculations became very cumbersome and numerically uncertain in the limit of very large PV masses.

In this paper, we want to treat gauge theories in order to check the preservation of gauge symmetry under the TLRS, directly at the physical dimension $D = 4$. In a gauge theory, the definition of fields presents some additional efforts. For charged fields, the translation operation in the distributional context has to be defined with due account of the the gauge environment, i.e. the associated gauge connection. For gauge-boson fields, the gauge transformation properties have to be implemented at the level of the distribution functional. In this gauge context, the first attempt using the OPVD concept concerned the analysis of QED anomalies [5]. It led very naturally to Fujikawa's analysis [6].

Dimensional regularization (DR) [7] is the generally adopted way of dealing with UV infinities in gauge theories, for it preserves gauge invariance. However, DR has frequent problems in the infrared (IR) where finite photon masses have to be introduced to eliminate IR singularities. It turns out that the TLRS provides the necessary mathematical formulation to avoid this kind of problem. Moreover, DR encounters additional difficulties with supersymmetric theories because invariance under supersymmetric transformations holds only for entire spacetime dimensions. To circumvent this problem, dimensional reduction was proposed [8], where the field components are unchanged in order to preserve supersymmetry. However, there arise ambiguities related to the treatment of the Lévi-Cività symbol $\epsilon^{\mu\nu\rho\sigma}$ and of γ^5. In addition, there can be complications with unitarity due to evanescent couplings when dimensional reduction is applied to non-supersymmetric theories as e.g. the standard model (SM).

In view of the physical importance of symmetry-conservation issues, it is very interesting, and even compelling, to compare our results with the TLRS to those, on the one hand, of DR and, on the other hand, of algebraic renormalization [9] based on the quantum action principle (QAP). Indeed the QAP allows one to control the breaking of symmetry induced by a non-invariant subtraction scheme such as the conventional Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) procedure [10]. In [2], it was shown that the TLRS formally includes BPHZ, which, however, owing to the specific properties of the PU-test function, operates in a non-conventional but a symmetry-conserving way.

It is our aim here to show that the TLRS can be applied directly at the physical dimension $D = 4$ to obtain typical quantities of gauge theories known either for their conventional IR divergent behaviour and/or violation of gauge symmetries. For example, the gauge-boson propagator with its particular problems in the LC gauge, the fermion self-energy and subsequent field renormalization fall in the first class. In the second class, one may cite the gauge-boson self-energy, Ward identities, BRST symmetry conservation in relation to the QAP treatment, etc. In section 2, we treat cases related to the first class, and those of the second class form the content of section 3. Finally, some conclusions and perspectives are presented in section 4.

2. Photon propagator with OPVD gauge fields in different gauges

2.1. OPVD gauge fields

An OPVD defines an operator functional with respect to a C^∞ test function $\rho(x^0, x) \in \mathcal{S}(\bar{\mathbb{R}}^4)$ (tensor product $\mathcal{S}(\mathbb{R})\,\hat{\otimes}\,\mathcal{S}(\mathbb{R}^3)$) the completed topological space of functions of fast decrease in the sense of Schwartz [11]. For the gauge field on a smooth flat manifold covered by a single coordinate system (chart), this operator functional can be written as

$$\mathbf{A}_\mu[\rho] \equiv \langle A_\mu, \rho\rangle = \int \mathrm{d}^4 y A_\mu(y)\rho(y). \tag{2.1}$$

The translated functional is a well-defined object [2], such that

$$\tau_x \mathbf{A}_\mu[\rho] = \langle \tau_x A_\mu, \rho\rangle = \langle A_\mu, \tau_{-x}\rho\rangle \equiv \mathbf{A}_\mu[\rho](x) \tag{2.2a}$$

$$= \int \mathrm{d}^4 y A_\mu(y)\rho(y - x). \tag{2.2b}$$

In the following, unless otherwise stated, for ease of notation, we shall write $\mathbf{A}_\mu(x)$ instead of $\mathbf{A}_\mu[\rho](x)$. Under a gauge transformation of the original $A_\mu(y)$, $\mathbf{A}_\mu(x)$ transforms as[4]

$$\mathbf{A}'_\mu(x) = \mathbf{A}_\mu(x) + \int \mathrm{d}^4y\, {}_y\partial_\mu[\Lambda(y)]\rho(x-y) \tag{2.3a}$$

$$= \mathbf{A}_\mu(x) + {}_x\partial_\mu \int \mathrm{d}^4y \Lambda(y)\rho(x-y) \tag{2.3b}$$

$$= \mathbf{A}_\mu(x) + \partial_\mu \mathbf{\Lambda}(x), \tag{2.3c}$$

where the last relation results from an integration by parts in y in the sense of distributions. $\mathbf{A}_\mu(x)$ is then taken as the physical field from which the Lagrangian and the propagator are constructed.

For the case of a non-flat metric, the concept of convolution can be treated in a chart-independent manner through trajectories with their associated tangent vectors. An example of this type is treated in appendix A.1, where the important role of convolution is emphasized whenever mathematical operations are dubious without test functions.

2.2. Abelian Lagrangian

In this subsection, we show that with the field $\mathbf{A}^\mu(x)$ of equation (2.2*a*) the Abelian gauge-invariant Lagrangian can be expressed in terms of physical (transverse) degrees of freedom only. We start with

$$\mathcal{L} = \frac{1}{2}\int \mathrm{d}^4x \mathbf{A}^\mu(x)[g_{\mu\nu}\Box - \partial_\mu\partial_\nu]\mathbf{A}^\nu(x) \tag{2.4a}$$

$$= -\frac{1}{2}\int \mathrm{d}^4k\, \mathcal{A}^\mu(k)k^2\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]\mathcal{A}^\nu(-k)f^2\big(k_0^2, \vec{\mathbf{k}}^2\big), \tag{2.4b}$$

where the pseudo-metric tensor is $g_{\mu\nu} \overset{\text{def}}{=} \text{diag}\{1,-1,-1,-1\}$, $\Box = \partial^\mu g_{\mu\nu}\partial^\nu$, $\mathcal{A}^\mu(k)$ is the Fourier transform of $A^\mu(x)$ and $f\big(k_0^2, \vec{\mathbf{k}}^2\big)$ is that of the test function $\rho(x)$ [2].

Remark 1. The expression $\mathcal{P}^T_{\mu\nu}(k) = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ is a non-invertible transverse projector $\big(\det\big[\mathcal{P}^T_{\mu\nu}(k)\big] = 0\big)$. It is well known that working in a specific gauge modifies this property and leads, by inversion, to a classical gauge-field propagator. For instance, the Lorentz gauge condition is implemented at the level of (2.4*a*) by the addition of a Lagrange-multiplier term $-\frac{1}{2\zeta}(\partial_\mu \mathbf{A}^\mu(x))^2$. Then, $\mathcal{P}^T_{\mu\nu}$ is no longer a projector and the determinant changes to $\det\big[g_{\mu\nu} - \frac{(\zeta-1)}{\zeta}\frac{k_\mu k_\nu}{k^2}\big] = -\frac{1}{\zeta} \neq 0$, thereby allowing inversion. The mathematically equivalent operation that changes $g_{\mu\nu}$ in $g_{\mu\nu}(1-\epsilon)$, with ϵ being arbitrary small, leads also to an invertible modification of the transverse operator $\mathcal{P}^T_{\mu\nu}(k)$. Clearly, in this way, the emphasis is put on the mathematical definition of the distributional inverse of k^2, with a link between ϵ and ζ (cf appendix A.2). This aspect shall be discussed further in the following.

However, in the linear functional approach, it is always possible to work with a transverse test function $\rho^{\text{tr}}_{\mu\nu}(x) = \big[P^T_{\mu,\nu}\rho\big](x) = \big(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\big)\rho(x)$, which still belongs to the space $\mathcal{S}$ of test functions ρ of rapid decrease.

Proposition 1. *With the transverse test function $\rho^{\text{tr}}_{\mu\nu}(x) = \big(g_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\big)\rho(x)$, the field $\mathbf{A}_\mu(x)$ in (2.2a) satisfies the Lorentz gauge condition $\partial_\mu \mathbf{A}^\mu(x) = 0$: it is purely transverse and gauge invariant. Then, the Lagrangian in (2.4a) can be written solely in terms of transverse degrees of freedom.*

[4] The usual symbol ∂_μ stands for $\frac{\partial}{\partial x^\mu}$; when the coordinate x^μ needs specification, we shall use instead the symbol ${}_x\partial_\mu$.

Proof. From (2.2*a*), and with transverse test functions as defined above, we have

$$\begin{aligned}\partial_\mu \mathbf{A}^\mu(x) &= \int \mathrm{d}^4 y A^\nu(y)\, {}_x\partial^\mu \left(g_{\nu\mu} - \frac{{}_y\partial_\nu\, {}_y\partial_\mu}{\Box}\right)\rho(y-x)\\ &\equiv \int \mathrm{d}^4 y A^\nu(y)\, {}_x\partial^\mu \left(g_{\nu\mu} - \frac{{}_x\partial_\nu\, {}_x\partial_\mu}{\Box}\right)\rho(y-x)\\ &= \int \mathrm{d}^4 y A^\nu(y)\, {}_x\partial_\nu \left(1 - \frac{{}_x\partial_\mu\, {}_x\partial^\mu}{\Box}\right)\rho(y-x) \equiv 0.\end{aligned}$$

The longitudinal projector $P^L_{\mu\nu}$ is such that: $\forall x\big[P^L_{\mu\nu} + P^T_{\mu\nu}\big](x) = g_{\mu\nu}$, then $P^L_{\mu\sigma}P^{T\,\sigma}{}_{\nu} = 0$ and

$$\begin{aligned}\mathbf{A}_\mu(x) &= \int \mathrm{d}^4 y A^\nu(y)\big[P^L_{\nu\sigma} + P^T_{\nu\sigma}\big](y)P^{T\,\sigma}{}_{\mu}(y)\rho(y-x)\\ &= \int \mathrm{d}^4 y \big[P^T_{\sigma\nu}A^\nu\big](y)P^{T\,\sigma}{}_{\mu}(y)\rho(y-x)\\ &\equiv \int \mathrm{d}^4 y A^T_\sigma(y)P^{T\,\sigma}{}_{\mu}(y)\rho(y-x) = \int \mathrm{d}^4 y A^T_\sigma(y)\rho^{tr\,\sigma}{}_{\mu}(y-x).\end{aligned}\tag{2.5}$$

The choice retained for $\rho^{\mathrm{tr}}_{\mu\nu}(x)$ is certainly not unique, for the projectors $P^{T,L}_{\mu\nu}$ are evidently gauge and chart dependent [17–21]. However, whatever the choice of $P^T_{\mu\nu}$, it is always possible to define $\rho^{\mathrm{tr}}_{\mu\nu}(x) = \big[P^T_{\mu,\nu}\rho\big](x)$ and the result (2.5) is general since the required sum $\big[P^L_{\mu\nu} + P^T_{\mu\nu}\big](x) = g_{\mu\nu}$ is gauge and chart independent. Thus, taking (2.5) as the physical field ensures working with physically relevant degrees of freedom only. Going back to (2.4*a*) we define the transverse test function in the Fourier space as $f^{\mathrm{tr}}_{\mu\nu}(k) = \mathcal{P}^T_{\mu\nu}(k)f(k_0^2, \vec{\mathbf{k}})$ (e.g. $\big[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\big]f(k_0^2, \vec{\mathbf{k}}^2)$ in the Lorentz gauge). It is easy to show that $f^{\mathrm{tr}}_{\mu\sigma}(k)f^{\mathrm{tr}\sigma}{}_{\nu}(k) \sim f^{\mathrm{tr}}_{\mu\nu}(k)$, for any product of PU's is an equivalent PU (cf appendix A of [2] for the meaning of the equivalence relation). Then, the Lagrangian can be written as

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\int \mathrm{d}^4 k\, \mathcal{A}^\mu(k)k^2 f^{\mathrm{tr}}_{\mu\sigma}(k)f^{\mathrm{tr}\sigma}{}_{\nu}(k)\mathcal{A}^\nu(-k)\\ &= -\frac{1}{2}\int \mathrm{d}^4 k\, \mathcal{A}^{T\,\mu}(k)k^2\mathcal{A}^T{}_{\mu}(-k)f^2(k_0^2, \vec{\mathbf{k}}^2).\end{aligned}\tag{2.6}$$

□

Remark 2. The generic form of the Lagrangian in (2.6) can only be used for a given choice of the projectors $P^{T,L}_{\mu\nu}$ fixing the polarization vectors of the gauge field. From the geometrical point of view, the space is the four-dimensional Minkowski spacetime $\mathcal{M}$ and this vector field lives in a fibre vector bundle $\Phi \xrightarrow{\Pi} \mathcal{M}$ having for its typical fibre the $U(1)$ gauge group $\mathcal{G}$. Through the choice of transverse test functions real physics takes place in the base space $\Phi/\mathcal{G}$. According to Vilkovisky–DeWitt [16] a specific choice of horizontal projection operator $\Pi_{\mu\nu}$ and connection exists, which lead to gauge theories without ghosts. For the flat metric used in (2.6), this horizontal projection operator in the Fourier space just reduces to our original $\mathcal{P}^T_{\mu\nu}(p)$, i.e. $\Pi_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$. We expect then our gauge formalism with transverse test functions to be also free of ghosts.

2.3. Gauge-boson propagator with OPVD gauge fields and the LC gauge

We now turn to a problem much debated in the literature on LC physics [17–20]: the form of the photon propagator in the LC gauge, in particular in the IR region.

For the purpose of comparison with standard results for the gauge-dependent propagator, we shall use here unprojected test functions. The propagator $\mathbf{G}_{\mu\nu}(x)$ can be defined equivalently (cf appendix B) by the time-ordered product of two $\mathbf{A}_\mu$, or by the convolution of the time-ordered product of two A_μ fields with two test functions[5] and the use of their Fourier transforms:

$$\mathbf{G}_{\mu\nu}(x) = -\imath\langle 0|T[\mathbf{A}_\mu(x)\mathbf{A}_\nu(0)]|0\rangle$$

$$\equiv -\imath\int \mathrm{d}^4z\,\mathrm{d}^4y'\langle 0|T[A_\mu(z+x)A_\nu(y')]|0\rangle\rho(z)\rho(y') \tag{2.7a}$$

$$= \int \mathrm{d}^4z\,\mathrm{d}^4y' G_{\mu\nu}(z+x-y')\rho(z)\rho(y') \tag{2.7b}$$

$$= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\exp[-\imath k\cdot x]}{k^2+\imath\epsilon}D_{\mu\nu}(k)f^2\big[k_0^2,\vec{k}^2\big], \tag{2.7c}$$

where the gauge-dependent quantity $D_{\mu\nu}(k)$ is given by

$$D_{\mu\nu}(k) = \sum_{\lambda=0}^{3}\epsilon_\mu^{(\lambda)}(k)\epsilon_\nu^{(\lambda)\star}(k) \tag{2.8a}$$

$$= -g_{\mu\nu} + (1-\zeta)\frac{k_\mu k_\nu}{k^2+\imath\epsilon}, \tag{2.8b}$$

with $\zeta = 1$ and $\zeta = 0$ corresponding respectively to the Feynman and Landau gauge. The LC gauge is one of the most frequently used gauge choices in perturbative QCD calculations and in non-perturbative light-front (LF) approaches. In the latter cases, besides the convenience already known from perturbative studies—built-in transversality of Green's functions, ghost-free procedure, etc—the LC gauge turns out to simplify greatly the treatment of constraints inherent to LC dynamics. However, difficulties and inconsistencies show up in usual LC quantization of gauge fields [17–19]. They have to do with specific spurious singularities appearing in the LC gauge-field propagator. The presence of the test function in (2.7c) naturally gives the proper mathematical treatment of these singularities according to the analysis of [2]. With n—($n^2 = 0$)—the original LC vector, the LC gauge is specified by the two conditions $n\cdot\mathbf{A} = 0$ and $\partial\cdot\mathbf{A} = 0$. In this case, $D_{\mu\nu}(k)$ is found to be [17]

$$D_{\mu\nu}(k) = -g_{\mu,\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{(n\cdot k)} - \frac{n_\mu n_\nu}{(n\cdot k)^2}k^2. \tag{2.9}$$

In the absence of test functions in (2.7c), the common wisdom is to treat the singularity in $\frac{1}{n\cdot k}$ with the Mandelstam–Leibbrandt (ML) prescription [20]. It is worthwhile to recall that such a prescription is compatible with the path integral formulation but not imposed by it. Different prescriptions for the spurious pole are possible depending on the gauge-field boundary conditions. However, the definition of higher powers of the spurious singularity is not settled by the ML prescription. Moreover—as observed in [19]—no complete regularization of the singularity is achieved when $n\cdot k$ and $n^*\cdot k \to 0$—($(n^*)^2 = 0, n\cdot n^* = 1$)—simultaneously and non-local UV divergent terms show up in loop diagrams with the ML prescription.

In terms of n and n^*, the measure is $\mathrm{d}^4k = d(n\cdot k)d(n^*\cdot k)\mathrm{d}^2\mathbf{k}_\perp$. After integration[6] over $k^- = n^*\cdot k$ in (2.7c), the following decomposition of $\mathbf{G}_{\mu\nu}$ at $D = 4$ is obtained:

$$\mathbf{G}_{\mu\nu}(x) = \frac{1}{(2\pi)^4}\{-\imath g_{\mu\nu}I_0(x) - (n_\nu\partial_\mu + n_\mu\partial_\nu)I_1(x) + n_\nu n_\mu\partial^2 I_2(x)\}, \tag{2.10a}$$

[5] Unless otherwise stated, test functions will be written as $f(p^2)$, although they belong to different topological spaces for Minkowskian and Euclidean varieties [2].

[6] The test function having no extension to the complex plane, contour techniques cannot be used here; instead, the method is that indicated in appendix E (equations (E.2) and (E.3)) of [2].

with

$$I_p(x) = -\imath\pi \int \mathrm{d}^2k_\perp \int_{-\infty}^{\infty} \frac{d(n\cdot k)}{(n\cdot k)^{p+1}} \mathrm{sign}(n\cdot k) \exp[-\imath k\cdot x] f^2\left[\frac{1}{2}\left(n\cdot k + \frac{k_\perp^2}{2n\cdot k}\right)^2\right]. \tag{2.10b}$$

Here, $\exp[-\imath k\cdot x] = \exp\left[-\imath\left(n\cdot kx^- + \frac{k_\perp^2}{2n\cdot k}x^+\right) + \imath k_\perp\cdot x_\perp\right]$, with $x^- = n^*\cdot x$ and $x^+ = n\cdot x$. $f^2\left[\frac{1}{2}\left(n\cdot k + \frac{k_\perp^2}{2n\cdot k}\right)^2\right]$ is a super regular test function (SRTF of Besov space type [17]) providing extension of distributions in $n\cdot k$ when either $n\cdot k \to 0$ or $nk \to \infty$. According to equation (4.16) of [2], these extensions (dubbed TLRS in what follows) are

$$\left[\widetilde{\frac{1}{(n\cdot k)^{p+1}}}\right] = \frac{(-)^p}{p!}\partial_{n\cdot k}^{p+1}\log[\mu(n\cdot k)] + 2\frac{(-)^p}{p!}H_p\delta^{(p)}(n\cdot k), \tag{2.11}$$

with $H_p = \gamma + \psi(p+1)$.

A test case is given by the I_0 contribution to $\mathbf{G}_{\mu\nu}$. With $a = \frac{k_\perp^2}{2}x^+ > 0$, $\quad b = x^- > 0$ and $z = \sqrt{\frac{b}{a}}(n\cdot k)$, the integral over $n\cdot k$ in (2.10b) for $p = 0$ can be obtained from the study of the general expression

$$I_{\mathbf{f}}(a,b) = \int_0^\infty \frac{\mathrm{d}z}{z}\exp\left[\imath\sqrt{ab}\left(z+\frac{1}{z}\right)\right]\mathbf{f}\left[\left(z+\frac{1}{z}\right)\right], \tag{2.12}$$

where $\mathbf{f}$ stands for the generic test function present in (2.10b). $I_{\mathbf{f}}(a,b)$ exists in the limit $\mathbf{f}\to 1$ over the whole integration domain, with the value

$$I_{\mathbf{f}=1}(a,b) = -\pi N_0(2\sqrt{ab}) + \imath\pi J_0(2\sqrt{ab}).$$

The limit $ab \to 0$ of $I_{\mathbf{f}=1}(a,b)$ (see below) can actually be obtained directly [22] from the integral itself without reference to the above exact result. The strategy is to expand $\exp\left[\imath\frac{\sqrt{ab}}{z}\right]$ and, at the point where the series does not formally exist, use the extension formula for $\left[\widetilde{\frac{1}{z^{p+1}}}\right]$ and resum. Then,

$$\begin{aligned}\lim_{\mathbf{f}\to 1} I_{\mathbf{f}}(a,b) &= \mathcal{N}\sum_{p=0}^{\infty}\frac{(-\imath\sqrt{ab})^p}{(p!)^2}\int_0^\infty \frac{\mathrm{d}z}{z}\exp[\imath\sqrt{ab}z]\left[\partial_z^{p+1}\log[\mu z] + 2H_p\delta^{(p)}(z)\right]\\ &= \mathcal{N}\sum_{p=0}^{\infty}\frac{(-\imath\sqrt{ab})^p}{(p!)^2}\underset{\epsilon\to 0}{\mathcal{L}}\left[\partial_z^{p+1}\log[\mu z] + 2H_p\delta^{(p)}(z); s\right].\end{aligned}$$

Here, $\mathcal{L}$ is the Laplace transform, in the sense of distributions, with $s = \epsilon - \imath\sqrt{ab}\,\Re(s) > 0$

$$\begin{aligned}\lim_{\mathbf{f}\to 1} I_{\mathbf{f}}(a,b) &= \mathcal{N}\sum_{p=0}^{\infty}\frac{(-1)^p(\sqrt{ab})^{2p}}{(p!)^2}\left[\log(\mu) - \log(\sqrt{ab}) + \imath\frac{\pi}{2} + \psi(p+1)\right]\\ &\underset{ab\to 0}{=} -2\log(\sqrt{ab}) - 2\gamma + \imath\pi = \mathcal{N}\left(\log(\mu) - \log(\sqrt{ab}) + \imath\frac{\pi}{2} - \gamma\right).\end{aligned}$$

Hence, $\mathcal{N} = 2$, $\mu = 1$. Regrouping different terms

$$\begin{aligned}\lim_{\mathbf{f}\to 1} I_{\mathbf{f}}(a,b) &= -2\sum_{p=0}^{\infty}\frac{(-1)^p(\sqrt{ab})^{2p}}{(p!)^2}[\log(\sqrt{ab}) + \gamma)] + 2\sum_{p=0}^{\infty}\frac{(-1)^p(\sqrt{ab})^{2p}}{(p!)^2}(\gamma + \psi(p+1))\\ &\quad + \imath\pi J_0(2\sqrt{ab})\\ &= -\pi N_0(2\sqrt{ab}) + \imath\pi J_0(2\sqrt{ab}).\end{aligned}$$

The method yields then a consistent distributional extension not only for the original LC singularities present in $\mathbf{G}_{\mu\nu}$ but also for any power of them, at variance with the ML prescription.

Another instructive comparison is given when considering the Fourier transforms G_{ML} and G_{TLRS} of the ML and distributional TLRS prescriptions $(p\cdot x = (n\cdot p)(n^*\cdot x) + (n^*\cdot p)(n\cdot x) - p_\perp\cdot x_\perp)$:

$$G_{\mathrm{ML}}(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\exp[\imath p\cdot x]}{n\cdot p + \imath\epsilon n^*\cdot p} = \frac{1}{2\pi}\frac{1}{n\cdot x - \imath\epsilon n^*\cdot x}\delta^2(x^\perp)$$

$$G_{\mathrm{TLRS}}(x) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\exp[\imath p\cdot x]\{2\partial_{n\cdot p}[\log(\mu \mid n\cdot p\mid)] + 4\underbrace{H_0}_{0}\delta(n\cdot p)\}$$

$$= \frac{\imath}{2\pi}(n^*\cdot x)\left[4\pi\gamma\delta(n^*x) + \pi\mathcal{P}f\left[\frac{\mathrm{sign}(n^*\cdot x)}{n^*\cdot x}\right]\right]\delta(n\cdot x)\delta^2(x^\perp)$$

$$= \frac{\imath}{2}\mathrm{sign}(n^*\cdot x)\delta(n\cdot x)\delta^2(x^\perp).$$

Note that both cases correspond to an inversion of $n\cdot\partial$ since $(n\cdot\partial)G_{\mathrm{TLRS}}(x) \equiv \frac{\partial}{\partial(n^*\cdot x)}G_{\mathrm{TLRS}}(x) = (n\cdot\partial)G_{ML}(x) = \imath\delta^4(x)$. The TLRS and ML prescriptions differ in the pseudo-function part present in the latter, a main characteristic of propagators in the causal approach [23].

3. Fermion and boson self-energies, and vertex functions with OPVD fields in dimension $D = 4$

The aim of this section is to apply the TLRS to various self-energy parts and vertices in order to

(1) compare the TLRS and DR UV results,
(2) show, in the case of IR-singularities, how the TLRS handles them successfully where DR has to introduce a non-zero gauge-boson mass,
(3) demonstrate the gauge-symmetry preserving nature of the TLRS,
(4) discuss the TLRS mechanism in relation to the QAP introduction of local non-invariant counterterms to restore broken symmetries induced by usual BPHZ subtractions.

Calculations of these quantities in the TLRS scheme with OPVD fields in the Feynman gauge $\zeta = 1$ can be found in [24]. We only recall here the main results in this gauge.

3.1. Fermion self-energy to one loop in the Feynman gauge

3.1.1. Comparison of TLRS and DR procedures. The TLRS fermionic self-energy $\Sigma^{(f)}(p) = A^{(f)}(p^2) + (\not p - m)B^{(f)}(p^2)$ is finite and can be written as

$$\Sigma^{(f)}(p) = \frac{e^2}{16\pi^2}\left[(-\not p + 4m)\log(\eta^2) + 2\int_0^1 \mathrm{d}x(\not p(1-x) - 2m)\log\left(x - \frac{p^2}{m^2}x(1-x)\right)f(p^2x^2)\right], \quad (3.1)$$

where the parameter η comes from the non-removable arbitrary scale present in any PU-test function, one remnant of which is $f(p^2x^2)$ [24]. This parameter η is directly related to the arbitrary DR mass scale μ by the relation $\eta^2 = \frac{4\pi\mu^2}{m^2}e^{-(\gamma+1)}$. While $\Sigma^{(f)}(p)$ in (3.1) is finite in the limit $f = 1$, the on-shell fermion field renormalization constant to one loop giving a unit residue at the pole of the propagator is[7] $Z_2^{(1,f)} = 2mA^{(f)'}(m^2) + B^{(f)}(m^2)$ and diverges if

[7] With $\Sigma_R^{(f)}(p) = \Sigma^{(f)}(p) - (\delta m^{(1,f)} + Z_2^{(1,f)}(\not p - m))$.

the limit $f \to 1$ is taken too early in the calculation of $\Sigma^{(f)}(p)$, as done in DR. Indeed one has in this case

$$A^{(f=1)'}(p^2) = \frac{e^2}{8\pi^2 m}\int_0^1 dx \frac{(1-x^2)}{1-\frac{p^2}{m^2}(1-x)},$$

where, for $p^2 = m^2$, the integrand diverges like $\frac{1}{x}$ when $x \to 0$. The usual practice is to give a 'small' mass λ to the photon such that

$$\begin{aligned}A^{(f=1)'}(m^2) &\underset{\lambda\to 0}{=} \frac{e^2}{8\pi^2 m}\int_0^1 dx \frac{x(1-x^2)}{x^2+\frac{\lambda^2}{m^2}(1-x)} \underset{\lambda\to 0}{=} \frac{e^2}{16\pi^2 m}\left[\int_{\frac{\lambda^2}{m^2}}^1 dx\frac{1}{x} - 1\right],\\ &\underset{\lambda\to 0}{=} \frac{e^2}{16\pi^2 m}\left[-1+\log\left(\frac{m^2}{\lambda^2}\right)\right] + \mathcal{O}(\lambda^2);\\ B^{(f=1)}(m^2) &= -\frac{e^2}{8\pi^2\epsilon} - \frac{e^2}{16\pi^2}\left[\log\left(\frac{4\pi\mu^2}{m^2}e^{-(\gamma+1)}\right)+3\right].\end{aligned}$$

For the TLRS expression, the test function $f(p^2x^2)$ is still there. One has instead[8] (argument of f is dimensionless)

$$\begin{aligned}A^{(f)'}(m^2) &= \frac{e^2}{8\pi^2 m}\int_0^1 dx\frac{(1-x^2)}{x}f(x^2) = \frac{e^2}{16\pi^2 m}\left[\int_0^1 dx\mathcal{P}f\left(\frac{1}{x}\right)-1\right],\\ &= \frac{e^2}{16\pi^2 m}\left[\lim_{\epsilon\to 0}\int_\epsilon^1 dx\frac{1}{x}+\log(\epsilon)-1\right] = -\frac{e^2}{16\pi^2 m},\\ B^{(f)}(m^2) &= -\frac{e^2}{16\pi^2}\log\eta^2 + \frac{e^2}{4\pi^2}\int_0^1 dx(1-x)\log x = -\frac{e^2}{16\pi^2}(\log\eta^2+3),\end{aligned}$$

where in the definition of the pseudo-function the scale factor is $a = 1$.

For $Z_2^{(1,f)}$, the respective results for DR and TLRS are then

$$Z_2^{(1,f=1)} = -\frac{e^2}{8\pi^2\epsilon} - \frac{e^2}{16\pi^2}\left[2\log\left(\frac{\lambda^2}{m^2}\right)+\log\left(\frac{4\pi\mu^2}{m^2}e^{-(\gamma+1)}\right)+5\right], \tag{3.2a}$$

$$Z_2^{(1,f)} = -\frac{e^2}{16\pi^2}(\log\eta^2+5). \tag{3.2b}$$

Hence, the TLRS-consistent mathematical extension of singular distributions both in the UV and IR regions leads to a finite fermi field renormalization and links in a transparent way[9] the arbitrary DR and PU scales μ and η.

3.1.2. Transcription of TLRS into BPHZ subtractions. As shown in [2, section 5], the TLRS can be mathematically transcribed into BPHZ subtractions, however not at zero external momentum but, due to scaling properties of PU-test functions, at an arbitrary momentum $q = \frac{p}{\eta^2}$. It is instructive to exhibit the BPHZ form of the TLRS self-energy discussed in the previous section. To be in line with the early general QAP analysis of [25, 26] to be discussed later, we shall use the parametric space representation of the self-energy $\Sigma^{(f)}(p)$. With the notation of [27], $\mathcal{T}$ labels the family (trees) of all connected proper subdiagrams associated

[8] No contribution to $A^{(f=1)'}(p^2)$ comes from $\partial_{p^2} f(p^2x^2)\mid_{p^2=m^2}$, for $\int_0^1 x^2(1+x)\log(x)f'(x^2)dx = 0$.

[9] Numerical constants in equations (3.2a) and (3.2b) have no physical significance as they can be absorbed in a rescaling of these arbitrary scales.

with each connected one-particle (1PI) irreducible diagram of a particular Feynman diagram G. Then, the Feynman amplitude takes the form

$$I_G(\mathbf{p}) = \int_0^\infty \prod_{i=1}^{I} \mathrm{d}\alpha_i \frac{\exp\left[-\left(\sum \alpha_l m_l^2 + \mathcal{Q}(\mathbf{p},\alpha)\right)\right]}{(4\pi)^{2L}\mathcal{P}(\alpha)^2},$$

where L is the number of independent loops, I is the number of internal particle lines, $\mathcal{Q}$ is a quadratic positive-definite form in the external momenta $\mathbf{p}$ and a homogeneous rational fraction of degree 1 in the α's. The polynomial $\mathcal{P}$ is a sum of monomial of degree L:

$$\mathcal{P}(\alpha) = \sum_{\text{trees}\,\mathcal{T}} \prod_{l\notin\mathcal{T}} \alpha_l.$$

If the superficial degree of divergence $\omega(g)$ for all subdiagram $g \in \mathcal{T}$ is negative, $I_G(\mathbf{p})$ is absolutely convergent. If not the usual BPHZ subtraction is that of [27] (cf section 8). It is based on the homogeneity properties of $\mathcal{Q}$ (respectively $\mathcal{P}$) under a simultaneous dilatation by a factor λ of the α's: $\mathcal{Q}(\mathbf{p},\lambda^2\alpha) = \mathcal{Q}(\lambda\mathbf{p},\alpha)$. Then, the kth-order Taylor subtractions at $\mathbf{p} = 0$ on $\mathcal{I}(\mathbf{p},\alpha) = \frac{\exp[\mathcal{Q}(\mathbf{p},\alpha)]}{\mathcal{P}^2(\alpha)}$ can be transcribed in terms of the result taken at $\lambda = 0$ of the action on $\mathcal{I}(\mathbf{p},\lambda^2\alpha)$ of a corresponding Taylor operator $\mathcal{F}^k$ in the dilation parameter λ. The essential property of this generalized Taylor expansion is that $(1-\mathcal{F}^k)\mathcal{I}(\mathbf{p},\alpha)$ is the Taylor remainder of $(k+1)$(th)-order in λ. Thereby, the original divergence in the integral on λ is removed and a finite Feynman amplitude $I_G^R(p)$ is obtained (cf equation (8.55) of [27]).

With this parametric α-representation, the electron self-energy to one loop $\Sigma^{(f=1)}(p)$ is given by equation (7.27a) of [27]. Then, in the Euclidean space ($p^2 = -\mathbf{p}^2$), the BPHZ subtraction at zero external momentum takes the form

$$\begin{aligned}(1-\mathcal{F}^1)\mathcal{I}(\mathbf{p},\lambda^2\alpha)\,|_{f=1} &= \frac{\exp[-\lambda\mathbf{p}^2x(1-x)]}{\lambda^2} - \frac{1}{\lambda^2}\sum_{s=0}^{1}\frac{\lambda^s}{s!}\frac{\mathrm{d}^s}{\mathrm{d}\lambda^s}[\exp(-\lambda\mathbf{p}^2x(1-x))]_{\lambda=0},\\ &\underset{\lambda\to0}{=} \frac{\mathbf{p}^4}{2}x^2(1-x)^2 + \mathcal{O}(\lambda).\end{aligned}$$

It is easy to check that this result is obtained simply with Lagrange's expression for the Taylor remainder $\mathcal{R}^{(1)}$ (cf [2, section 3]):

$$\frac{1}{\lambda^2}\mathcal{R}^{(1)}[\lambda^2\mathcal{I}(\mathbf{p},\lambda^2\alpha)\,|_{f=1}] = -\int_1^\infty \frac{\mathrm{d}t}{t^2}(1-t)\partial^2_{(\lambda)}\left\{\exp\left[-\frac{\lambda}{t}\mathbf{p}^2x(1-x)\right]\right\}. \qquad (3.3)$$

Here, no arbitrary scale appears, at variance with the TLRS and DR on-shell renormalization procedures. However, a scale can be introduced by a BPHZ subtraction at an arbitrary momentum. Of course, all different procedures can be related by the introduction of finite counterterms. But schemes apparently convenient in practice often violate some symmetries and their resaturation imposes a definite choice of finite counterterms. Both DR and BPHZ at zero-momentum subtraction encounter problems with chiral and BRST invariances [9, 28, 29], to say nothing about the complicated treatment of IR-divergences [30].

With test functions, equation (3.3) takes a form similar to that of equations (C.9)–(C.11) in [2] with respective SRTF test functions of dimensionless arguments $f\left[\frac{y}{u(1-x)}\right]$ and $f\left[\frac{y}{ux}\right]$, where $u = \lambda m^2$. In the two different x-sectors $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$, the product of $\mathcal{I}(\mathbf{p},\lambda^2\alpha)$ with the relevant test function is equal to its Taylor remainder of arbitrary order (1 in this case), which can be written with Lagrange's formula. Every integral being now finite[10] the overall result is that of equation (3.3) with however an upper limit $\tilde{\eta}^2$ imposed by running-boundary

[10] See for example, however, the treatment of field renormalization below where a remnant part $f(p^2x^2)$ of the initial test functions is still present.

properties of PU-test functions when extended to unity over the whole integration domain in λ:

$$\frac{1}{\lambda^2}\mathcal{R}^{(1)}[\lambda^2\mathcal{I}(\mathbf{p},\lambda^2\alpha)\ |_f] \underset{\lim f\to 1}{=} -\int_1^{\tilde{\eta}^2}\frac{\mathrm{d}t}{t}(1-t)\partial^2_{(\lambda)}\left\{\exp\left[-\frac{\lambda}{t}\mathbf{p}^2x(1-x)\right]\right\},$$
$$\underset{\lambda\to 0}{=}\frac{1}{2}x^2(1-x)^2\left(\mathbf{p}^2-\frac{\mathbf{p}^2}{\tilde{\eta}^2}\right)^2+\mathcal{O}(\lambda).$$

Clearly, this corresponds[11] to $\frac{1}{\lambda^2}\mathcal{R}^{(1)}[\lambda^2(\mathcal{I}(\mathbf{p},\lambda^2\alpha)\ |_{f=1}\ -\mathcal{I}(\mathbf{q},\lambda^2\alpha))\ |_{f=1}]_{\mathbf{q}^2=\frac{\mathbf{p}^2}{\eta^2}}$, which is the standard BPHZ subtraction at momentum $\mathbf{q}=\frac{\mathbf{p}}{\eta}$, as shown in [2, section 5]. We shall discuss the importance of this result in the following in relation to the conservation of gauge symmetry.

3.1.3. Gauge-dependent contributions to the fermion self-energy with OPVD to one loop. We turn now to the gauge contributions to the self-energy not discussed in [24]. Their origin is in the second term of equation (2.8*b*).

Proposition 2. *The arbitrary scale a in the definition of the pseudo-function is in direct correspondence with the gauge choice parameter* ζ *by the relation* $a=\exp[-(\frac{\zeta\log\zeta}{1-\zeta}+1)]$.

Proof. To one loop, these gauge-dependent contributions can be written as [27]

$$\Sigma^{(f)}_\zeta(p)\underset{\lambda\to 0}{=}\imath e^2(1-\zeta)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\not{k}(\not{p}-\not{k}+m)\not{k}}{(p-k)^2-m^2}\frac{f(k^2)f((p-k)^2)}{(k^2-\lambda^2+\imath\epsilon)(k^2-\zeta\lambda^2+\imath\epsilon)},$$
$$=\imath e^2(1-\zeta)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{\not{k}(p^2-m^2)-k^2(\not{p}-m)}{(p-k)^2-m^2}-\not{k}\right]\frac{f(k^2)f((p-k)^2)}{k^4},$$
$$=\imath e^2(1-\zeta)\left(I_1^f-I_2^f-I_3^f\right);$$
$$I_1^f=2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{(1-x)(\not{k}+\not{p}x)(p^2-m^2)}{[k^2-m^2x+p^2x(1-x)]^3}f(k^2)f(p^2x^2);$$
$$I_2^f=2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{(1-x)(k+px)^2(\not{p}-m)}{[k^2-m^2x+p^2x(1-x)]^3}f(k^2)f(p^2x^2).$$

In the second line, due to the test function $f(k^2)$, it is legitimate to take directly the limit $\lambda\to 0$, whereas in DR all the f's are just unity from the start and a finite photon mass λ is needed to avoid IR divergences. By symmetry, $I_3^f=0$ and, since $I_1^f|_{\not{p}=m}=I_2^f|_{\not{p}=m}=0$, I_1^f and I_2^f do not contribute to a shift of the fermion mass but do contribute to the fermi field renormalization constant to one-loop $Z_2^{(1)}$. In effect, one has then

$$Z^{(1),f}_{2,\zeta}=\imath e^2(1-\zeta)\frac{\partial\left(I_1^f-I_2^f\right)}{\partial\not{p}}\bigg|_{\not{p}=m}.$$

The overall DR ($f=1$) conventional result for $Z^{(1),f=1}_{2,\zeta}$ is well known [32] and can be written as

$$Z^{(1),f=1}_{2,\zeta}=\frac{e^2}{(4\pi)^2}(1-\zeta)\left\{\frac{2}{\epsilon}-\log\left(\frac{\lambda^2}{m^2}\right)+\log\left(\frac{4\pi\mu^2}{m^2}e^{-(\gamma+1)}\right)+\frac{\zeta\log\zeta}{1-\zeta}+2\right\},\qquad(3.4)$$

[11] With the irrelevant modification $\eta^2=\frac{\tilde{\eta}^2}{\sqrt{2\tilde{\eta}^2-1}}$.

and for the TLRS, we have

$$\frac{\partial(I_1)}{\partial \not p}\Big|_{\not p=m} = -\frac{2\imath}{(4\pi)^2}\int_0^1 \frac{\mathrm{d}x}{x}(1-x)f(x^2) = -\frac{2\imath}{(4\pi)^2}\left[\frac{1}{2}\int_0^1 \mathrm{d}u\mathcal{P}f\left(\frac{1}{u}\right)-1\right],$$
$$= -\frac{2\imath}{(4\pi)^2}\left[\lim_{\epsilon\to 0}\frac{1}{2}\left(\int_{a\epsilon}^1 \frac{\mathrm{d}u}{u}+\log\epsilon\right)-1\right] = \frac{2\imath}{(4\pi)^2}\left[\frac{1}{2}\log a + 1\right],$$

where we use the scale arbitrariness in the definition of the pseudo-function distribution. We have next (cf appendix C1 for details)

$$\frac{\partial(I_2)}{\partial \not p}\Big|_{\not p=m} = 2\int_0^1 \mathrm{d}x(1-x)\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k^2+m^2x^2)}{[k^2-m^2x^2]^3}f\big(k_0^2,\mathbf{k}^2\big)f(m^2x^2),$$
$$= \frac{\imath}{16\pi^2}[\log\eta^2+3]. \tag{3.5}$$

The overall TLRS result for $Z_{2,\zeta}^{(1,f)}$ can be written as

$$Z_{2,\zeta}^{(1,f)} = \frac{e^2}{(4\pi)^2}(1-\zeta)[\log\eta^2-\log a+1]. \tag{3.6}$$

Upon inspection of equations (3.4) and (3.6), the finite part of $Z_{2,\zeta}^{(1,f=1)}$ and $Z_{2,\zeta}^{(1,f)}$ are equal with the identifications:

(ii) $\eta^2 = \frac{4\pi\mu^2}{m^2}e^{-(\gamma+1)}$ already encountered,
(ii) $a = \exp\left[-\left(\frac{\zeta\log\zeta}{1-\zeta}+1\right)\right]$.

It is seen that $\lim_{\zeta\to 1} a = 1$, in agreement with the definition of the pseudo-function distribution used earlier in the calculation of $Z_2^{(1,f)}$ in the Feynman gauge. □

3.2. *Gauge-boson vacuum polarization to one loop with OPVD gauge fields*

3.2.1. *Lorentz structure of vacuum polarization with the TLRS and DR.*

Proposition 3. *The Lorentz structure of the vacuum polarization emerges naturally in the TLRS, directly at the physical dimension $D = 4$, with no UV/IR divergences.*

Proof. The boson vacuum polarization to one loop with PU-test functions can be written as [32]

$$\Pi_{\mu,\nu}^{(1,f)}(p) = -4\imath e^2\int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^4k}{(2\pi)^4}$$
$$\times\frac{2x(1-x)(p_\mu p_\nu - g_{\mu\nu}p^2)+g_{\mu\nu}\left(\frac{k^2}{2}-m^2+p^2x(1-x)\right)}{\big[k^2-(\underbrace{m^2-p^2x(1-x)}_{m^2(x,p^2)})\big]^2}f(k^2). \tag{3.7}$$

□

We evaluate first the two contributions from the longitudinal term, with the same procedure in the Euclidean space [1, 2] used in equation (3.5), with notation $\mathbf{p}\equiv p_E$. The results can be written (cf appendix C2 for details) as

$$\frac{1}{2}\int\frac{\mathrm{d}^4\mathbf{k}}{(2\pi)^4}\frac{\mathbf{k}^2f(\mathbf{k}^2)}{[\mathbf{k}^2+m^2(x,\mathbf{p}^2)]^2} = -\frac{m^2(x,\mathbf{p}^2)}{(4\pi)^2}\log\left[\frac{\eta^2m^2}{m^2(x,\mathbf{p}^2)}\right], \tag{3.8a}$$

$$m^2(x,\mathbf{p}^2)\int\frac{\mathrm{d}^4\mathbf{k}}{(2\pi)^4}\frac{f(\mathbf{k}^2)}{[\mathbf{k}^2+m^2(x,\mathbf{p}^2)]^2} = \frac{m^2(x,\mathbf{p}^2)}{(4\pi)^2}\log\left[\frac{\eta^2m^2}{m^2(x,\mathbf{p}^2)}\right]. \tag{3.8b}$$

The sum is zero: $\Pi^{(1,f)}_{\mu,\nu}(p)$ is transverse at $D = 4$. It is interesting to note that, at variance with a straight cut-off procedure, there is no quadratically diverging contribution in the second term, for, with a PU-test function, such a term is now

$$\int_0^\infty \mathrm{d}(k^2) f(k^2) = \int_0^\infty \mathrm{d}X f(X) \equiv \int_0^\infty \frac{\mathrm{d}X}{X^2} f\left(\frac{1}{X^2}\right), \tag{3.9a}$$

$$= \int_0^\infty \mathrm{d}X Pf\left(\frac{1}{X^2}\right) = \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{\mathrm{d}X}{X^2} - \frac{1}{\epsilon} = 0! \tag{3.9b}$$

It is instructive to recall that in DR there is an *axiomatic* consistency requirement, which states that [33]

$$\forall\alpha(>0 \text{ or } <0) \int \mathrm{d}^D\mathbf{k}(\mathbf{k}^2)^\alpha = 0,$$

and the same sum of longitudinal contributions at arbitrary D reduces to

$$(4\pi)^{-\frac{D}{2}}(m^2(x,\mathbf{p}^2))^{(\frac{D}{2}-1)}\left\{-\Gamma\left[2-\frac{D}{2}\right]+\Gamma\left[2-\frac{D}{2}\right]\right\} = 0.$$

The final vacuum polarization to one loop can be written as $\Pi^{(1,f)}_{\mu,\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi^{(1,f)}(p^2) \equiv p^2 d_{\mu\nu}\bar{\omega}^{(1,f)}(p^2)$ with

$$\Pi^{(1,f)}(p^2) = 8\imath e^2\int_0^1 \mathrm{d}x x(1-x)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{f(k^2)}{[k^2-m^2(x,p^2)]^2}.$$

The respective TLRS and DR ($f = 1$) contributions are

$$\Pi^{(1,f)}(p^2) = -\frac{e^2}{2\pi^2}\left\{\frac{1}{6}\log\eta^2 - \int_0^1 \mathrm{d}x x(1-x)\log\left[\frac{m^2(x,p^2)}{m^2}\right]\right\}, \tag{3.10a}$$

and

$$\Pi^{(1,f=1)}(p^2) = -\frac{e^2}{2\pi^2}\left\{\frac{1}{3\epsilon}+\frac{1}{6}\log\left[\frac{4\pi\mu^2 e^{-(\gamma+1)}}{m^2}\right]\right.$$
$$\left. - \int_0^1 \mathrm{d}x x(1-x)\log\left(\frac{m^2(x,p^2)}{m^2}\right) + \mathcal{O}(\epsilon)\right\}. \tag{3.10b}$$

As is well known, to recover Coulomb's law at a large distance $\bar{\omega}^{(1,f)}(p^2)$ needs renormalization with the condition $\bar{\omega}^{(1,f)}_R(0) = 0$. It is commonly performed by an on-shell subtraction $\bar{\omega}^{(1,f)}_R(p^2) = \bar{\omega}^{(1,f)}(p^2) - \bar{\omega}^{(1,f)}(0)$, involving only finite terms at $D = 4$ for the TLRS but, for DR at $D = 4-\epsilon$, subtraction of extra diverging terms when $\epsilon \to 0$ are also necessary.

3.2.2. Problems of vacuum polarization with BPHZ. In parametric representation, the term violating gauge symmetry may be written as $g_{\mu,\nu}\Delta\bar{\omega}^{(f=1)}(\mathbf{p}^2)$ with (cf [27, section 7.1.1])

$$\Delta\bar{\omega}^{(f=1)}(\mathbf{p}^2) = \frac{e^2}{4\pi^2}\int_0^1 \mathrm{d}x\int_0^\infty\frac{\mathrm{d}\lambda}{\lambda^2}[1+\lambda m^2(\mathbf{p}^2,x)]\exp[-\lambda m^2(\mathbf{p}^2,x)],$$

$$= -\frac{e^2}{4\pi^2}\int_0^1 \mathrm{d}x\int_0^\infty \mathrm{d}\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\left[\frac{1}{\lambda}\exp[-\lambda m^2(\mathbf{p}^2,x)]\right]\right\}.$$

The λ-integral diverges as λ^{-1} when $\lambda \to 0$ and the same BPHZ subtraction needed to validate Coulomb's law is $\Delta\bar{\omega}^{(f=1)}(\mathbf{p}^2) - \Delta\bar{\omega}^{(f=1)}(0)$. It leads to a renormalized finite value $\Delta\bar{\omega}^{(f=1)}_R(p^2) = -\frac{e^2}{24\pi^2}\mathbf{p}^2$. However, the subtraction procedure is only defined up to a first-order polynomial in $\mathbf{p}^2$ and a specific non-symmetric finite counterterm

$\Delta_{\rm CT}(\mathbf{p}^2)$ proportional to $\mathbf{p}^2$ will act to restore the gauge symmetry. Clearly, its value is $\Delta_{\rm CT}(\mathbf{p}^2) = -\int_0^1 {\rm d}x\mathbf{p}^2[\frac{\rm d}{{\rm d}\mathbf{p}^2}\exp[-\mathbf{p}^2x(1-x)]_{\mathbf{p}^2=0}$ so that the overall subtraction is $\Delta\bar{\omega}^{(f=1)}(\mathbf{p}^2) - \Delta\bar{\omega}^{(f=1)}(0) + \Delta_{\rm CT}(\mathbf{p}^2)$, which just corresponds to the use of the Taylor remainder of first order according to equation (3.3).

With a PU-test function of dimensionless argument, it is useful to extract the mass dimension of $\Delta\bar{\omega}^{(f)}(\mathbf{p}^2)$ and use dimensionless variable in every integrals. Then,

$$\Delta\bar{\omega}^{(f)}(\mathbf{p}^2) = \frac{e^2m^2}{4\pi^2}\int_0^1 {\rm d}x \int_0^\infty \frac{{\rm d}\lambda}{\lambda^2}\left(1+\lambda\frac{m^2(\mathbf{p}^2,x)}{m^2}\right) \times \exp\left[-\lambda\frac{m^2(\mathbf{p}^2,x)}{m^2}\right]\int_0^\infty {\rm d}u u e^{-u} f\left(\frac{u}{\lambda}\right). \tag{3.11}$$

In keeping with the earlier use of (3.3), the SRTF PU $f(\frac{u}{\lambda})$ regulates explicitly the integral over λ through its first-order Taylor remainder (cf [2, equation (C.11)]):

$$f\left(\frac{u}{\lambda}\right) \equiv -\lambda^2\int_1^\infty \frac{{\rm d}t}{t}(1-t)\partial_\lambda^2 f\left(\frac{ut}{\lambda}\right).$$

If used as before under the form of a BPHZ subtraction or as such with successive integration by parts in λ, one obtains $\Delta\bar{\omega}_R^{(f)}(p^2) = 0$. Clearly, the presence of the test functions dictates the use of Lagrange's formula for the Taylor remainder, since the term-by-term symmetry-violating BPHZ decomposition at zero momentum is avoided.

3.3. Vertex function and Ward–Takahashi identity to one loop with OPVD

The Ward–Takahashi (WT) identity [32] relates the electron–photon vertex function to the inverse of the complete fermionic propagator $\mathbb{S}_F^{-1}(p)$, i.e. $\Gamma_\mu(p,q,p+q) = \mathbb{S}_F^{-1}(p+q) - \mathbb{S}_F^{-1}(p)$ and $\Gamma_\mu(p,q,p+q) =_{q\to 0} \frac{\partial\mathbb{S}_F^{-1}(p)}{\partial p^\mu}$. To lowest order in the QED coupling e, the free fermion propagator is $S_F^{-1}(p) = \not{p} - m$ and $\frac{\partial S_F^{-1}(p)}{\partial p^\mu} = \Gamma_\mu^{(0)}(p,0,p) = \gamma_\mu$. Hence, the vertex function is written as $\Gamma_\mu(p,q,p+q) = \gamma_\mu + \Lambda_\mu(p,q,p+q)$ with $\Lambda_\mu(p,0,p) = -\frac{\partial\Sigma(p)}{\partial p^\mu}$, for $\mathbb{S}_F^{-1}(p) = S_F^{-1}(p) - \Sigma(p)$.

Remark 3. As there is no contribution to $\frac{\partial\Sigma(p)}{\partial p^\mu}$ (cf footnote 8) from the derivative of the test function $f(p^2x^2)$ in equation (3.1), the derivation of the Ward identity to one loop with OPVD is the standard one [32]. As a consequence, the renormalization constant $Z_1^{(1,f)}$ to one loop of the vertex counterterm contribution $-{\rm i}eZ_1^{(1,f)}\gamma_\mu$ is equal to $Z_2^{(1,f)}$. With results of earlier sections, this useful and direct validation exercise with the TLRS on-shell is exposed in appendix B.

3.4. SU(3) gauge theory to one loop with OPVD

The quark self-energy is simply the corresponding QED expression times a group theoretic factor [32]. The gluon self-energy is more interesting in that there are four one-loop contributions to calculate. As the gluon–quark one-loop contribution is simply the corresponding QED expression times another group theoretic factor, only the gluon–ghost and three- and four-gluon one-loop contributions need specific TLRS calculations. The four-gluon term does not contribute since the loop is proportional to $\int {\rm d}^4k\frac{f(k^2)}{k^2}$ evaluated to zero in equations (3.9*a*) and (3.9*b*). The three-gluon one-loop contribution can be written as [27, 32]

$${}^{3G}\Pi_{\mu\nu}^{ab,(1,f)}(\mathbf{p}) = \imath\frac{g^2}{2}\underbrace{f^{acd}f^{bcd}}_{C_A\delta^{ab}}\int\frac{{\rm d}^4\mathbf{k}}{(2\pi)^4}\frac{N_{\mu\nu}(\mathbf{k},\mathbf{p})}{\mathbf{k}^2(\mathbf{k}+\mathbf{p})^2}f(\mathbf{k}^2)f((\mathbf{k}+\mathbf{p})^2),$$

where f^{abc} are the structure constants of the group, $C_A = 3$ for $SU(3)$ and

$$N_{\mu\nu}(k,p) = \delta^{\mu\nu}(5p^2 + 2k\cdot p + 2k^2) + 5(k_\mu p_\nu + k_\nu p_\mu + 2k_\mu k_\nu) - 2p_\mu p_\nu.$$

In turn, the gluon–ghost one-loop term is

$$^{Gg}\Pi^{ab,(1,f)}_{\mu\nu}(\mathbf{p}) = -\imath g^2 C_A \delta^{ab} \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{(\mathbf{k}+\mathbf{p})_\mu \mathbf{k}_\nu}{\mathbf{k}^2(\mathbf{k}+\mathbf{p})^2} f(\mathbf{k}^2) f((\mathbf{k}+\mathbf{p})^2).$$

Noting that in the UV regime the test functions behave like $f^2(\mathbf{k}^2) \sim f(\mathbf{k}^2)$, the sum of these two contributions is obtained in the usual way and can be written as

$$^{3G}\Pi^{ab,(1,f)}_{\mu\nu}(\mathbf{p}) + ^{Gg}\Pi^{ab,(1,f)}_{\mu\nu}(\mathbf{p}) = \imath \frac{g^2}{2} C_A \delta^{ab} \int_0^1 dx \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{\mathcal{N}_{\mu\nu}(\mathbf{k},\mathbf{p},x)}{[\mathbf{k}^2 + \mathbf{p}^2 x(1-x)]^2} f(\mathbf{k}^2),$$

with

$$\mathcal{N}_{\mu\nu}(k,p,x) = 2(1+4x(1-x))(\delta_{\mu\nu}p^2 - p_\mu p_\nu) + \delta_{\mu\nu}(4k^2 + 3p^2 - 10p^2 x(1-x)).$$

We show below that the longitudinal contribution vanishes. It is useful to recall first the DR result for this term [27]

$$^{L}I^{ab,(1,f=1)}_{\mu\nu}(\mathbf{p}) = \imath \frac{g^2}{2} C_A \delta^{ab} \delta_{\mu\nu} \frac{(D-1)\Gamma(2-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}(D-2)(p^2)^{(1-\frac{D}{2})}} \times \left[(D-2)B\left(\frac{D}{2}-1, \frac{D}{2}-1\right) - 4(D-1)B\left(\frac{D}{2}, \frac{D}{2}\right)\right],$$

with $\quad B(\alpha,\beta) = \int_0^1 dx x^{(\alpha-1)}(1-x)^{(\beta-1)} = \dfrac{\Gamma[\alpha]\Gamma[\beta]}{\Gamma[\alpha+\beta]}$.

The overall result is zero because of the identity

$$(D-2)B\left(\frac{D}{2}-1, \frac{D}{2}-1\right) = 4(D-1)B\left(\frac{D}{2}, \frac{D}{2}\right).$$

Note that for $D = 4$ it is just the relation $2\int_0^1 dx\ (1-6x(1-x)) = 0$.

With the TLRS, the derivation is in two steps. Observe first that since $\int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{f(\mathbf{k}^2)}{\mathbf{k}^2} = 0$, we also have

$$\int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{f(\mathbf{k}^2)}{\mathbf{k}^2} \equiv \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{f(\mathbf{k}^2)(\mathbf{k}+\mathbf{p})^2}{\mathbf{k}^2(\mathbf{k}+\mathbf{p})^2} = \int_0^1 dx \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{\mathbf{k}^2 + \mathbf{p}^2(1-x)^2}{[\mathbf{k}^2 + \mathbf{p}^2 x(1-x)]^2} f(\mathbf{k}^2) = 0,$$

i.e.

$$\int_0^1 dx \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{\mathbf{k}^2 f(\mathbf{k}^2)}{[\mathbf{k}^2 + \mathbf{p}^2 x(1-x)]^2} = -\frac{\mathbf{p}^2}{2} \int_0^1 dx(1-2x(1-x)) \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{f(\mathbf{k}^2)}{[\mathbf{k}^2 + \mathbf{p}^2 x(1-x)]^2}.$$

The next step is to collect terms for the longitudinal contribution $\delta_{\mu\nu}(4p^2 + 3p^2 - 10p^2 x(1-x))$. We end up with the evaluation of

$$^{L}I^{ab,(1,f)}_{\mu\nu}(\mathbf{p}) = \imath \frac{g^2}{2} C_A \delta^{ab} \delta_{\mu\nu} \mathbf{p}^2 \int_0^1 dx(1-6x(1-x)) \int \frac{d^4\mathbf{k}}{(2\pi)^4} \frac{f(\mathbf{k}^2)}{[\mathbf{k}^2 + \mathbf{p}^2 x(1-x)]^2}, \quad (3.12a)$$

$$\equiv \imath \frac{g^2}{2} C_A \delta^{ab} \delta_{\mu\nu} \mathbf{p}^2 \int_0^1 dx(1-6x(1-x)) J(\mathbf{p}^2, x) = 0, \quad (3.12b)$$

since $J(\mathbf{p}^2, x)$ is actually independent of x (cf appendix C.3 for details). The overall TLRS two-point function for $\mathcal{A}_a^\mu(p)\mathcal{A}_b^\nu(-p)$ is therefore finite and is just the DR result given in equation (12-112) of [27] without the divergence in $\frac{2}{\epsilon}$ and $\log \frac{p^2}{\mu^2}$ replaced by the TLRS expression $\log \frac{p^2}{\Lambda^2} - \log \eta^2$.

We shall not dwell further on other superficially divergent proper functions as they lead to the same observations. The analysis in terms of BPHZ subtractions violating gauge and BRST symmetries to one loop and their resaturation by finite and symmetry-violating counterterms follows that of section (3.2.2) for QED. However, the general derivation of Slavnov–Taylor identities calls for a proper definition of path integrals in the sense of [15] and is beyond the scope of our present study.

4. Conclusions

In this paper, we applied to gauge theories a recently published renormalization technique, based on the extension of singular distributions, which we called the Taylor Lagrange renormalization scheme (TLRS). The purpose was to show the method at work on examples usually treated with DR and BPHZ subtractions improved in relation to the QAP, well known from the literature. In all cases considered, we were able to demonstrate that the TLRS method respects gauge invariance. In addition, we showed how the gauge degree of freedom is related to the scale arbitrariness in the definition of pseudo-functions occurring in the IR treatment of zero-mass propagators.

As to the comparison with DR, the main results are as follows.

(a) In the UV, the TLRS finite expressions coincide with the finite parts of their $\overline{\text{MS}}$-counterparts in DR and a direct relation is established between the DR arbitrary mass scale and the scale inherent to the definition of the test functions. Terms involving this latter scale are essential for renormalization group studies and is the only remaining fingerprint of the test functions. Technically, the TLRS, through the definition of regular distributions, eliminates the $\frac{1}{\epsilon}$ divergent parts of DR and trades the mass scale dependence of the coupling for the scale dependence of the test functions.
(b) The elimination of divergences is intimately related to the suppression of causality violating terms in initial propagators.
(c) A similar result is obtained in QCD for the three-gluon one-loop contribution to the gluon propagator.
(d) Again for QED, in the IR, there are no divergence problems due to the zero mass of the photon, contrary to DR where in many cases a finite photon mass has to be kept in order to avoid divergences.

As to the comparison with BPHZ subtractions at zero external momentum known to violate symmetries, our conclusions are as follows.

(a) The TLRS includes indeed BPHZ as stated in [2] with subtractions not at zero external momentum but at the external momentum scaled by the inherent scaling property of PU-test functions.
(b) These BPHZ subtractions are encoded in a single expression known as Lagrange's integral formula for the corresponding Taylor remainder. It is only in the presence of PU-test functions that this integral formula acquires a specific domain of integration leading to suppression of initial symmetry-violating divergences of physical amplitudes.
(c) The net effect of this subtraction is in conformity with algebraic renormalization and the QAP that introduces non-symmetric counterterms to compensate for the symmetry

violation of conventional BPHZ. In this sense, it is the symmetry-improved BPHZ, which is contained in the TLRS.

As another example, we presented the case of the photon propagator in LC gauge. The spurious singularities arising in this gauge are most frequently treated by the widely accepted but otherwise ad hoc ML prescription. In contrast to ML, the TLRS method, in keeping with the mathematically funded IR analysis, yields a finite distributional extension not only of the original LC singularity but also of all powers of it.

Some important aspects of TLRS have yet to be examined such as its implementation in chiral theories (cf however [5] in relation to [28]) and BRST symmetries in the context of Slavnov–Taylor identities. Their derivation usually proceeds through functional integrals, which, in the presence of test functions, demand special mathematical treatments in the line of [15], far beyond the scope of this presentation.

Acknowledgments

We acknowledge financial support from CNRS/IN2P3-Department during the course of this work. EW is grateful to Alain Falvard and Fabrice Feinstein for their kind hospitality at the LPTA and LUPM.

Appendix A. Illustrative examples

A.1. Electric current associated with a charged point particle

In Minkowski space $\mathcal{M}$ with metric $\mathbf{g}$, the current is characterized by contravariant distribution components, which define translated functionals with respect to a C^∞ test function[12] $\rho(x^0, \mathbf{x})$ in the sense of (2.2*a*)

$$\begin{aligned}\mathbf{J}_\mu[\rho](x) &= \int \mathrm{d}^4 y J_\mu(y)\rho(y-x) = \int \mathrm{d}^4 y\left[q\int \delta^4(y-x(\tau))\frac{\mathrm{d}y_\mu}{\mathrm{d}\tau}\,\mathrm{d}\tau\right]\rho(y-x)\\ &= q\int \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\rho(x-x(\tau))\,\mathrm{d}\tau \equiv q\int \sqrt{-g(x)}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\frac{1}{\sqrt{-g(x)}}\rho(x-x(\tau))\,\mathrm{d}\tau,\end{aligned}$$

where $g(x) = \det[g_{\alpha\beta}(x)] < 0$ and the invariant parametrization of the particle trajectory in terms of the proper time τ is used. We note that if $(\rho_j)_{j\in\mathbb{N}}$ is a sequence of Dirac functions (e.g. $x \mapsto \rho_j(x) = \frac{j^4}{\pi^2}e^{-j^2x_0^2}e^{-j^2r^2}$ with $r^2 = \Sigma_{i=1}^3(x^i)^2$), then, when $j \to \infty$, the sequence converges to the reflection symmetric $\delta(x^0)\delta^3(\mathbf{x}) \equiv \delta^4(x)$. Hence, $\frac{1}{\sqrt{-g(x)}}\delta^4(x - x(\tau))$ is just the covariant coordinate expression for the scalar δ-function: $\mathbf{J}^\mu[\rho](x)$ transforms covariantly, vanishes on the boundary of a domain centred on the world line of the particle and is valid in any coordinate system [12]. Integrating by parts as in (2.3*b*), it is easy to check current conservation, i.e. $\frac{1}{\sqrt{-g(x)}}\partial^\mu[\sqrt{-g(x)}\mathbf{J}_\mu[\rho](x)] = 0$ (or in terms of the codifferential $\boldsymbol{\delta}$ of exterior calculus in differential geometry [12] $\boldsymbol{\delta}\mathbf{J}[\rho] = 0$).

This example introduces convolution in a chart-independent manner through trajectories with their associated tangent vectors. Every vector field defines an ensemble of trajectories or *flow* ϕ_t which in turn leads to the notion of transport of functions by this flow [13]. For instance, on $\mathbb{R}$, a constant vector field with component $V_x = 1$, i.e. $V = \frac{\mathrm{d}}{\mathrm{d}x}$, generates a flow such that

$$V = \frac{\mathrm{d}}{\mathrm{d}x} \quad \Leftrightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 \quad \Leftrightarrow \phi_t : x(t) = x(0) + t.$$

[12] For ease of notation, test functions will be written as $\rho(x)$, bearing in mind that they belong to different topological spaces for Minkowskian and Euclidean varieties [2].

Then, the function f_0 at $t = 0$ transported by the flow is given by

$$f_t : \mathbb{R} \ni x \mapsto f_t(x) = f_0(\phi_{-t}(x)) = f_0(x - t).$$

A.2. Gauge-invariant phase factor of arbitrary loops in two-dimensional Abelian theory

Let γ be a closed loop[13] around the point x in two-dimensional Euclidean spacetime and denote its winding number around x by $\theta(\gamma, x)$. It is given by a contour integral in the complex plane over the path Γ of the loop

$$\begin{aligned}\theta(\gamma, x) &= \frac{1}{2\imath\pi}\int_\Gamma \frac{\mathrm{d}z}{z-x}\\ &\equiv -\frac{1}{2\pi}\epsilon_{\mu\nu}\oint \frac{\mathrm{d}z^\mu (z-x)^\nu}{|z-x|^2} = \oint \mathrm{d}z^\mu b_\mu(z,x),\end{aligned}$$

where $\epsilon_{\mu\nu}$ $\{(\mu,\nu) = 1,2,\ \epsilon_{1,2} = -\epsilon_{2,1} = 1\}$ is the two-dimensional antisymmetric tensor and

$$b_\mu(z,x) = -\frac{1}{2\pi}\epsilon_{\mu\nu}\frac{(z-x)^\nu}{|z-x|^2} = -\frac{1}{4\pi}\epsilon_{\mu\nu}\,{}_z\partial^\nu \log[M^2|z-x|^2], \tag{A.1}$$

where, for dimensional reason, M is an arbitrary scale parameter. Then, $b_\mu(z,x)$ may be viewed as the 'gauge potential' at z due to a magnetic vortex of unit strength located at x [14]. However, even though the antisymmetry of the tensor $\epsilon_{\mu\nu}$ would seem to infer the Lorentz gauge condition ${}_z\partial^\mu b_\mu(z,x) = 0$, this is not the case with the presence of a cut in the complex plane with the branch point at the position of the vortex. The exact direction of the cut is irrelevant, for its change is just a gauge transformation of $b_\mu(z,x) \to b_\mu(z,x) + \partial_\mu\Lambda(z)$ and θ itself is gauge invariant since $\oint \mathrm{d}z^\mu\partial_\mu\Lambda(z) = 0$. However, a remark is in order here: in (A.1), $b_\mu(z,x)$ and its expression in terms of ${}_z\partial^\nu \log[\Lambda^2|z-x|^2]$ have to be considered in the sense of *distributions*—it is in this framework that the equality between the two expressions for $b_\mu(z,x)$ acquires a precise mathematical meaning in the presence of the test function ρ. To be specific, in the sense of Fourier transforms of distributions, we have

$$b_\mu(z,x) = \imath\epsilon_{\mu\nu}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{k^\nu}{k^2}e^{\imath k(z-x)}$$

and due to the reflection symmetry $\rho(x) = \rho(-x)$

$$\rho(x) = \int \frac{\mathrm{d}^2k}{(2\pi)^2}e^{\imath kx}f(k^2).$$

Then, the convoluted product $\mathbf{b}_\mu[\rho](x,z)$ takes the form

$$\begin{aligned}\mathbf{b}_\mu[\rho](z,x) &= \int \mathrm{d}^2y\, b_\mu(y,x)\rho(y-z) = \imath\epsilon_{\mu\nu}\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{k^\nu}{k^2}e^{\imath k(z-x)}f(k^2),\\ &= \epsilon_{\mu\nu}{}_z\partial^\nu\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{k^2}e^{\imath k(z-x)}f(k^2),\\ &= \epsilon_{\mu\nu}{}_z\partial^\nu\left\langle \frac{e^{\imath k(z-x)}}{k^2}, f(k^2)\right\rangle = \epsilon_{\mu\nu}{}_z\partial^\nu\left\langle \frac{e^{\imath k(z-x)}}{\widetilde{k^2}}, 1\right\rangle,\end{aligned}$$

where in the second line the position of the partial derivative ${}_z\partial^\nu$ is legitimate, for the integral is well defined due to the presence of the Fourier-space test function $f(k^2)$. With $f(k^2)$ as a PU with implicit dimensionless argument $k^2 \curvearrowright \frac{k^2}{\Lambda^2} = X$, $\frac{1}{\widetilde{k^2}}$ is obtained from equation (4.16) of

[13] The loop is supposed not to intersect and overlap with itself.

[2] (cf also [1]) with the order index $n = 0$ and $H_0 = 0$. Then, $\frac{1}{k^2} = \partial_{k^2} \log\left(\frac{k^2}{\Lambda^2}\right) \doteq \mathcal{P}f\left(\frac{1}{k^2}\right)$ is the extended *pseudo-function* distribution [11] defined on $\mathcal{S}'(\mathbb{R})$ of the distribution $\frac{1}{k^2}$ initially defined on $\mathcal{S}'(\mathbb{R}\backslash\{0\})$. After the mathematical operations leading to $\mathcal{P}f\left(\frac{1}{k^2}\right)$, this PU can be extended to 1 over the whole domain of integration as indicated in the last line of the expression for $\mathbf{b}_\mu[\rho](z, x)$. Taking the Fourier transform of this pseudo-function distribution gives

$$\epsilon_{\mu\nu\,z}\partial^\nu \int \frac{d^2k}{(2\pi)^2} \mathcal{P}f\left(\frac{e^{\imath k(z-x)}}{k^2}\right) = -\epsilon_{\mu\nu\,z}\partial^\nu \frac{1}{2\pi}\left\{\gamma_E + \log\left[\frac{M|x-z|}{2}\right]\right\}, \tag{A.2}$$

where γ_E is Euler's constant. This is the result (A.1) as it should. However, the exercise is not purely academic, for it shows two important pieces of information:

(i) the role of convolution whenever mathematical operations are dubious without test functions,

(ii) the scale arbitrariness inherent to the definition of the pseudo-function (cf [11, chapter 2, p 41]) is tantamount to a freedom in the gauge choice for $\mathbf{b}_\mu[\rho](z, x)$.

Indeed $\forall a > 0 \in \mathbb{R} \int_0^a dx \mathcal{P}f(\frac{1}{x}) \doteq \lim_{\epsilon\to 0} \int_\epsilon^a \frac{dx}{x} + \log \epsilon = \log(a) \equiv \lim_{\epsilon'\to 0} \int_{\epsilon'}^1 \frac{dx}{x} + \log \epsilon'$ with $\epsilon' = a\epsilon$. In (A.2), $a = 1$ and for $a = \exp[f(z)]$, with $\partial^\nu f(z) = \epsilon^{\nu\sigma}\partial_\sigma \mathbf{\Lambda}(z)$ and $\epsilon_{\mu\nu}\epsilon^{\nu\sigma} = \delta_\mu^\sigma$, the term $\partial_\mu \mathbf{\Lambda}(z)$ is added.

From there on it is easy to find that $\epsilon_{\mu\nu\,x}\partial^\nu \theta(\gamma, x) = J_\mu(x)$ and for $\mathbf{A}_\mu(x)$ and a closed loop γ parametrized by s, $0 \leqslant s \leqslant 1$, we shall have

$$\begin{aligned}\int_\Gamma dz_\mu \mathbf{A}^\mu(z) &= \int_0^1 ds \frac{dz_\mu(s)}{ds} \mathbf{A}^\mu(z(s)),\\ &= \int_{\Sigma_\gamma} d^2z J_\mu(z)\mathbf{A}^\mu(z) = \int_{\Sigma_\gamma} d^2z \epsilon_{\mu\nu\,z}[\partial^\nu \theta(\gamma, z)]\mathbf{A}^\mu(z),\\ &= -\int_{\Sigma_\gamma} d^2z\theta(\gamma, z)\epsilon_{\mu\nu\,z}\partial^\nu \mathbf{A}^\mu(z) = \frac{1}{2}\int_{\Sigma_\gamma} d^2z\theta(\gamma, z)\epsilon_{\mu\nu}\mathbf{F}^{\mu\nu}(z),\end{aligned}$$

where Σ_γ is the area enclosed by the loop γ. In two dimensions, $\mathbf{F}_{\mu\nu}(z) = \epsilon_{\mu\nu}\mathbb{F}(z)$ and we have the well-known result that the gauge-invariant Abelian phase factor $\mathcal{U}(\gamma)$ is given in terms of the winding number of closed loops by

$$\mathcal{U}(\gamma) = \exp\left[\imath e \int d^2z\theta(\gamma, z)\mathbb{F}(z)\right].$$

Note that the integral is well defined, for $\mathbb{F}(z)$ inherits from convolution the topological properties attached to the test function ρ [2]. The functionals $\mathbf{A}_\mu[\rho]$ and $\mathbb{F}[\rho]$ being linear in ρ, the final evaluation of the vacuum expectation value (VEV) $\langle\mathcal{U}(\gamma)\rangle$ is that of a standard functional Gaussian integral in $\mathbb{F}$ with a *bona fide* integration measure $\Pi_x d\mathbb{F}[\rho](x)$[15].

Appendix B. Abelian gauge-boson propagator with test functions

With the the following conventions:

$$d\Omega_k = \frac{d^3\mathbf{k}}{(2\pi)^3 2k_0};\ k_0 = |\vec{k}|;\ \left[a_{(\lambda)}(k), a^\dagger_{(\lambda')}(k')\right] = (2\pi)^3 2k_0 \delta_{\lambda\lambda'}\delta^3(\vec{\mathbf{k}} - \vec{\mathbf{k}}'),$$

the original $A_\mu(x)$ takes the usual quantized form

$$A_\mu(x) = \int d\Omega_k \sum_{\lambda=0}^{3} \left[a_{(\lambda)}(k)\epsilon_\mu^{(\lambda)}(k)e^{(-ik\cdot x)} + a^\dagger_{(\lambda)}(k)\epsilon_\mu^{(\lambda)\star}(k)e^{(ik\cdot x)}\right]. \tag{B.1}$$

Now the inversion symmetric test function $\rho(x-y)$ has a well-defined Fourier decomposition

$$\rho(x-y) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} e^{\mathrm{i}q(x-y)} f\big(q_0^2, \vec{\mathbf{q}}^2\big). \tag{B.2}$$

It follows from (2.2*a*) that

$$\mathbf{A}_\mu(x) = \int d\Omega_k \sum_{\lambda=0}^{3} \big[a_{(\lambda)}(k)\epsilon_\mu^{(\lambda)}(k)e^{(-\mathrm{i}k\cdot x)} + a_{(\lambda)}^\dagger(k)\epsilon_\mu^{(\lambda)\star}(k)e^{(\mathrm{i}k\cdot x)}\big] f\big(k_0^2, \vec{\mathbf{k}}^2\big). \tag{B.3}$$

Due to the presence of the test function $\mathbf{A}_\mu(x)$ is a regular OPVD and we can perform the usual time ordering operation T by multiplying products of fields by step functions. With the usual gauge-dependent notation $D_{\mu\nu}(k) = \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(k)\epsilon_\nu^{(\lambda)\star}(k)$, one obtains

$$\begin{aligned}\mathbf{G}_{\mu\nu}(x) &= -\imath\langle 0|T[\mathbf{A}_\mu(x)\mathbf{A}_\nu(0)]|0\rangle, \\ &= -\int \frac{\mathrm{d}^3\mathbf{k}}{2|\mathbf{k}|(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \big[e^{-\imath\omega x_0 + \imath k\cdot x} + \text{c.c.}\big] \frac{D_{\mu\nu}(k)}{\omega - \imath\epsilon} f^2\big(k_0^2, \vec{\mathbf{k}}^2\big)\end{aligned} \tag{B.4}$$

$$= -\int \frac{\mathrm{d}^3\mathbf{k} dk_0}{2|\mathbf{k}|(2\pi)^4} \left[\frac{e^{-\imath k\cdot x}}{|\mathbf{k}| - k_0 - \imath\epsilon} + \frac{e^{-\imath k\cdot x}}{|\mathbf{k}| + k_0 - \imath\epsilon}\right] D_{\mu\nu}(k) f^2(\hat{k}^2, \vec{\mathbf{k}}^2). \tag{B.5}$$

In (B.3) and (B.4), the argument k_0^2 of the test function is the on-shell value $\hat{k}^2 = |\mathbf{k}|^2$. For $x^2 \neq 0$, the integral over $\vec{k}$ in (B.4) converges without the test function, which can then be taken to 1 on the integration domain. On the other hand, for $x^2 = 0$, (B.4) diverges without the test function. The treatment of this LC singularity is carried out as discussed in section 4 and appendix E of [2]. In the test function, it amounts to use the integration variable k_0 in place of the on-shell value $\hat{k}$. It follows then that

$$\begin{aligned}\mathbf{G}_{\mu\nu}(x) &= -\int \frac{\mathrm{d}^3\mathbf{k}\mathbf{dk_0}}{2|\mathbf{k}|(2\pi)^4} \left[\frac{e^{-\imath k\cdot x}}{|\mathbf{k}| - k_0 - \imath\epsilon} + \frac{e^{-\imath k\cdot x}}{|\mathbf{k}| + k_0 - \imath\epsilon}\right] D_{\mu\nu}(k) f^2\big(k_0^2, \vec{\mathbf{k}}^2\big) \\ &= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-\imath k\cdot x}}{k^2 + \imath\epsilon} D_{\mu\nu}(k) f^2\big(k_0^2, \vec{\mathbf{k}}^2\big).\end{aligned} \tag{B.6}$$

On the other hand, we have the standard result

$$\langle 0|T[A_\mu(z+x)A_\nu(y')]|0\rangle = \imath \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-\imath k\cdot(z+x-y')}}{k^2 + \imath\epsilon} D_{\mu\nu}(k), \tag{B.7}$$

and with the Fourier transform (B.2), after integration on z and y', one obtains

$$-\imath \int \mathrm{d}^4 z\, \mathrm{d}^4 y' \langle 0|T[A_\mu(z+x)A_\nu(y')]|0\rangle \rho(z)\rho(y') = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{-\imath k\cdot x}}{k^2 + \imath\epsilon} D_{\mu\nu}(k) f^2\big(k_0^2, \vec{\mathbf{k}}^2\big),$$

which establishes the equivalence between the two alternative expressions for $\mathbf{G}_{\mu\nu}(x)$.

Appendix C. TLRS on-shell electron–photon vertex to one loop

In the Feynman gauge, the DR($D = 4 - \epsilon$) expression for the electron–photon vertex to one loop $\Lambda^{(1,f=1)}$ is well known [27, 32] and can be written as

$$\begin{aligned}\Lambda_\mu^{(1,f=1)}(q, q') &= -\imath e^2 \mu^\epsilon \int \frac{\mathrm{d}^D p}{(2\pi)^D} \gamma^\alpha \frac{\not p + \not q' + m}{(p+q')^2 - m^2} \gamma_\mu \frac{\not p + \not q + m}{(p+q)^2 - m^2} \gamma_\alpha \frac{1}{p^2 - \lambda^2}, \\ &= -2\imath e^2 \mu^\epsilon \int \mathrm{d}x\, \mathrm{d}y \theta(1-x-y) \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{N_\mu(p, q, q')}{\big(p^2 - D\big(x, y, \Delta_q^2\big)\big)^3},\end{aligned} \tag{C.1}$$

where m is the bare electron mass, λ is the photon mass (to handle IR divergences) and the γ's are Dirac matrices, $\Delta_q = q' - q$. The numerator can be written as

$$N_\mu = \gamma_\mu \left\{ \frac{(2-D)^2}{D} p^2 - {\Delta_q}^2[2(1-x)(1-y) - (4-D)xy] + m^2[4(1-x-y) + (2-D)(x+y)^2] \right\} - \imath m \sigma_{\mu\nu} {\Delta_q}^\nu (x+y)(2 + (2-D)(x+y)),$$

$$\text{with} \quad \sigma_{\mu\nu} = \frac{\imath}{2}[\gamma_\mu, \gamma_\nu], \quad D(x, y, {\Delta_q}^2) = m^2(x+y)^2 + \lambda^2(1-x-y) - {\Delta_q}^2 xy.$$

Only the term in p^2 from N_μ is UV divergent. Let its name be $I_{\rm UV}^{(f)}({\Delta_q}^2)$. The corresponding expression with the TLRS involves the test functions $f((p+q'(1-y)-xq)^2)$, $f((p-yq'+q(1-x))^2)$ and $f((p-yq'-xq)^2)$. When $\lambda = 0$, the denominator has a pole when $p = x = y = 0$, but this pole is taken care of by $f((p-yq'-xq)^2)$. To examine the UV behaviour, we shall first keep λ finite and in this case all the test functions behave like $f^3(p^2) \sim f(p^2)$ (cf appendix A of [2]). With DR, $I_{\rm UV}^{(f=1)}({\Delta_q}^2)$ can be written as

$$\begin{aligned} I_{\rm UV}^{(f=1)}({\Delta_q}^2) &= 2e^2\mu^\epsilon \frac{(2-D)^2}{D} \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \int \frac{{\rm d}^D\mathbf{p}}{(2\pi)^D} \frac{\mathbf{p}^2}{[\mathbf{p}^2 + D(x, y, {\Delta_q}^2)]^3}, \\ &= \frac{e^2}{8\pi^2\epsilon} + \frac{e^2}{8\pi^2}\left[\frac{1}{2}\log\left(\frac{4\pi\mu^2 e^{-(\gamma+1)}}{m^2}\right) - \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \log\left(\frac{D(x, y, {\Delta_\mathbf{q}}^2)}{m^2}\right)\right]. \end{aligned} \quad \text{(C.2)}$$

Then, the renormalized UV-finite expression is

$$I_{{\rm UV},R}^{(f=1)}({\Delta_q}^2) = I_{\rm UV}^{(f=1)}({\Delta_q}^2) - I_{\rm UV}^{(f=1)}({\Delta_q}^2 = 0).$$

With the TLRS, we have

$$\begin{aligned} I_{\rm UV}^{(f)}({\Delta_q}^2) &= 2e^2 \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \int \frac{{\rm d}^4\mathbf{p}}{(2\pi)^4} \frac{\mathbf{p}^2 f(\mathbf{p}^2)}{\left[\mathbf{p}^2 + D(x, y, {\Delta_\mathbf{q}}^2)\right]^3}, \\ &= \frac{e^2}{8\pi^2} \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \int_0^\infty {\rm d}X \frac{X^2 f(X)}{\left[X + \frac{D(x,y,{\Delta_\mathbf{q}}^2)}{m^2}\right]^3}, \\ &= \frac{e^2}{8\pi^2} \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \int_0^\infty {\rm d}Y \partial_Y \left[\frac{Y^3}{(Y+1)^3}\right] \int_{\frac{D(x,y,{\Delta_\mathbf{q}}^2)}{m^2}}^{\eta^2} \frac{{\rm d}t}{t} f(Yt), \quad \text{(C.3)} \\ &= \frac{e^2}{8\pi^2}\left[\frac{1}{2}\log\eta^2 - \int_0^1 {\rm d}x \int_0^{1-x} {\rm d}y \log\left(\frac{D(x, y, {\Delta_\mathbf{q}}^2)}{m^2}\right)\right]. \quad \text{(C.4)} \end{aligned}$$

In (C.3), the integral over Y is self-converging without the PU-test function giving (C.4) in the limit $f(Yt) \to 1$ over the whole integration domain on Y. Comparing (C.4) with (C.2), DR and TLRS scales are linked in the same manner already encountered.

Although (C.4) is UV-finite, it is still IR-diverging in the limit $\lambda \to 0$. It is best seen when writing $\Lambda_{\mu,R}^{(1,f=1)}(q, q')$ as

$$\Lambda_{\mu,R}^{(1,f=1)}(q, q') = \gamma_\mu F_1\left({\Delta_q}^2\right) + \frac{\imath}{2m}\sigma_{\mu\nu}{\Delta_q}^\nu F_2\left({\Delta_q}^2\right),$$

where to one loop [27] and with $\sinh^2\frac{\theta}{2} = -\frac{{\Delta_q}^2}{4m^2}$.

$$F_1\left({\Delta_q}^2\right) \underset{\lambda\to 0}{=} \frac{e^2}{4\pi^2}\left[\left(\frac{1}{2}\log\frac{\lambda^2}{m^2} + 1\right)(\theta\coth\theta - 1) - 2\coth\theta \int_0^{\theta/2} {\rm d}\varphi\,\varphi\tanh\varphi - \frac{\theta}{4}\tanh\frac{\theta}{2}\right].$$

Actually, $F_1(\Delta_q{}^2)$ can be decomposed as [27]

$$F_1(\Delta_q{}^2) = \Sigma_{j=1}^4 \mathcal{F}_j + \text{constant} \quad \text{with } F_1(0) = 0,$$

and only $\mathcal{F}_1$ is IR divergent when $\lambda \to 0$, with the following expression:

$$\mathcal{F}_1 \underset{\lambda\to 0}{=} \frac{e^2}{4\pi^2}\theta \coth\theta \log\frac{\lambda}{m} - \frac{e^2}{2\pi^2}\coth\theta\int_0^{\theta/2} \mathrm{d}\varphi\varphi\tanh\varphi. \tag{C.5}$$

Our concern is then the fate of this IR divergence with the TLRS. When $\lambda = 0$, this divergence has its origin in the pole of the denominator in equation (C.1) occurring when $\{p, x, y\} = 0$. As noticed, in the UV regime, the product of the three test functions reduces to $f^3(p^2)$, whereas in the IR it reduces to $f((q'(1-y)-xq)^2) f((yq'-q(1-x))^2) f((yq'+xq)^2)$. The arguments of the first two are non-zero and the limit $f \to 1$ is liable for them while the third one regulates the diverging IR behaviour. It is then legitimate to use the following reduction of the product of the f's as $f^2(p^2) f((yq'+xq)^2) \sim f(p^2) f((yq'+xq)^2)$, valid both in the UV and IR regimes. For the TLRS, the accompanying IR-test function has the on-shell dimensionless argument

$$\frac{(yq'+xq)^2}{m^2} = x^2 + y^2 + 2xy\frac{q\cdot q'}{m^2} = (x+y)^2 - xy\frac{\Delta_q{}^2}{m^2}.$$

With $\Delta_q{}^2 = -4m^2\sinh^2\frac{\theta}{2}$, this test function can be written as

$$f\left[(x+y)^2 - xy\frac{\Delta_q{}^2}{m^2}\right] = f(x^2+y^2+2xy\cosh\theta)$$

and one has

$$\begin{aligned}\mathcal{F}_1(\theta) &= -\frac{e^2}{4\pi^2}\cosh\theta\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{f(x^2+y^2+2xy\cosh\theta)}{x^2+y^2+2xy\cosh\theta},\\ &= -\frac{e^2}{4\pi^2}\cosh\theta\int_0^1\mathrm{d}u\int_0^1\frac{\mathrm{d}s}{s}\frac{f(s^2(u^2+(1-u)^2+2u(1-u)\cosh\theta)}{(u^2+(1-u)^2+2u(1-u)\cosh\theta)},\end{aligned}$$

and with $D(u,\theta) = u^2 + (1-u)^2 + 2u(1-u)\cosh\theta$, this can be written as

$$\begin{aligned}\mathcal{F}_1(\theta) &= -\frac{e^2}{4\pi^2}\cosh\theta\int_0^1\frac{\mathrm{d}u}{D(u,\theta)}\frac{1}{2}\int_0^{D(u,\theta)}\frac{\mathrm{d}X}{X}f(X),\\ &= -\frac{e^2}{8\pi^2}\cosh\theta\int_0^1\frac{\mathrm{d}u}{D(u,\theta)}\int_0^{D(u,\theta)}\mathrm{d}X\mathcal{P}f\left(\frac{1}{X}\right),\\ &= -\frac{e^2}{8\pi^2}\cosh\theta\int_0^1\frac{\mathrm{d}u}{D(u,\theta)}\left[\lim_{\epsilon\to0}\int_\epsilon^{D(u,\theta)}\frac{\mathrm{d}X}{X}+\log\epsilon\right],\\ &= -\frac{e^2}{8\pi^2}\cosh\theta\int_0^1\mathrm{d}u\frac{\log D(u,\theta)}{D(u,\theta)},\end{aligned}$$

and with $(1-2u)\tanh\frac{\theta}{2} = \tanh\varphi$, it finally reduces to

$$\mathcal{F}_1(\theta) = -\frac{e^2}{2\pi^2}\coth\theta\int_0^{\theta/2}\mathrm{d}\varphi\varphi\tanh\varphi,$$

which is (C.5) without the IR-diverging term in $\log\frac{\lambda}{m}$.

To complete the proof of total finiteness of the electron–photon vertex function to one loop with the TLRS, we envisage now the on-shell gauge contribution to $\Lambda_\mu^{(1)}(q, q')$. Its origin comes from the gauge contribution to the photon propagator (cf subsection (3.2)) and can be written as

$$\Lambda_{\zeta,\mu}^{(1,f)} = -e^2(1-\zeta)\gamma_\mu\int\frac{\mathrm{d}^4\mathbf{p}}{(2\pi)^4}\frac{f(\mathbf{p}^2)}{(\mathbf{p}^2+\lambda^2)(\mathbf{p}^2+\zeta\lambda^2)}.$$

In DR, with $D = 4 - \epsilon$, $f = 1$, the integral measure is $\frac{d^D p}{(2\pi)^D}$, $e^2 \curvearrowright e^2\mu^\epsilon$ and the photon mass has to be kept finite. The gauge contribution in this case can be written as

$$\Lambda^{(1,f=1)}_{\zeta,\mu} = -\frac{e^2}{16\pi^2}(1-\zeta)\gamma_\mu\left[\frac{2}{\epsilon} + \log\left(\frac{4\pi\mu^2 e^{-(\gamma+1)}}{m^2}\right) + \log\frac{m^2}{\lambda^2} + \frac{\zeta}{1-\zeta}\log\zeta + 1\right]. \tag{C.6}$$

With the TLRS, due to the presence of the test function, the photon mass λ can be set to zero and one obtains

$$\Lambda^{(1,f)}_{\zeta,\mu} = -e^2(1-\zeta)\gamma_\mu \int \frac{d^4\mathbf{p}}{(2\pi)^4}\frac{f(\mathbf{p}^2)}{\mathbf{p}^4} = -\frac{e^2}{16\pi^2}(1-\zeta)\gamma_\mu \int_0^\infty \frac{dX}{X} f(X),$$
$$= -\frac{e^2}{16\pi^2}(1-\zeta)\gamma_\mu\left[\int_0^\Lambda \frac{dX}{X} f(X) + \int_0^{\frac{1}{\Lambda}} \frac{dX}{X} f\left(\frac{1}{X}\right)\right]. \tag{C.7}$$

Here, both $f(X)$ and $f(\frac{1}{X})$ have the necessary properties [2] leading to the pseudo-function distribution extension $\mathcal{P}f(\frac{1}{X})$ of $\frac{1}{X}$ at the origin, however, with different arbitrary scalings, one, a, coming from the initial IR-region, and another, η^2, coming from the original UV region transformed by a change of integration variable $X \curvearrowright \frac{1}{X}$. Thus, (C.7) becomes

$$\Lambda^{(1,f)}_{\zeta,\mu} = -\frac{e^2}{16\pi^2}(1-\zeta)\gamma_\mu \lim_{\epsilon\to 0}\left[\left(\int_{a\epsilon}^\Lambda \frac{dX}{X} + \int_{\frac{\epsilon}{\eta^2}}^{\frac{1}{\Lambda}} \frac{dX}{X}\right) + 2\log\epsilon\right],$$
$$= -\frac{e^2}{16\pi^2}(1-\zeta)\gamma_\mu(\log\eta^2 - \log a). \tag{C.8}$$

Comparing (C.8) and the finite part of (C.6), we reach the same identifications as those of subsection (3.2):

(i) $\eta^2 = \frac{4\pi\mu^2}{m^2}e^{-(\gamma+1)}$ already encountered,
(ii) $a = \exp\left[-\left(\frac{\zeta\log\zeta}{1-\zeta} + 1\right)\right]$.

Complete coherence, as it should from remark 3, is therefore established between finite parts of the QED self-energy and vertex function in DR and TLRS.

Appendix D. TLRS calculations of some four-dimensional integrals

D.1. Integrals occurring in relation (6)

We look for evaluation of

$$\frac{\partial(I_2)}{\partial \not{p}}\Big|_{\not{p}=m} = 2\lim_{\epsilon\to 0}\int_\epsilon^1 dX(1-x)\int \frac{d^4k}{(2\pi)^4}\frac{(k^2+m^2x^2)}{[k^2-m^2x^2]^3} f(k_0^2, \vec{\mathbf{k}}^2) f(m^2x^2). \tag{D.1}$$

Taking into account appendix E of [2], with $\mathbf{k} = k_E$ and test functions of dimensionless arguments appropriate to an Euclidean variety $\left(f(\mathbf{k}^2) \equiv f\left(\frac{\mathbf{k}^2}{m^2}\right), f(m^2x^2) \equiv f(x^2)\right)$, this relation transforms to

$$\frac{\partial(I_2)}{\partial \not{p}}\Big|_{\not{p}=m} = -2\imath \lim_{\epsilon\to 0}\int_\epsilon^1 dX(1-x)\int \frac{d^4\mathbf{k}}{(2\pi)^4}\frac{(-\mathbf{k}^2+m^2x^2)}{[\mathbf{k}^2+m^2x^2]^3} f(\mathbf{k}^2) f(m^2x^2),$$

and with the change $(m^2x^2 > 0)$ $\mathbf{k}^2 = m^2Yx^2$, one obtains

$$\frac{\partial(I_2)}{\partial \not{p}}\Big|_{\not{p}=m} = -\frac{\imath}{8\pi^2}\lim_{\epsilon\to 0}\int_\epsilon^1 dX(1-x)\int_0^\infty Y\,dY\frac{(-Y+1)}{(Y+1)^3}\underbrace{f(Yx^2)}_{-\int_1^\infty dt\,\partial_t\, f(Yx^2t)} f(x^2),$$

but $-\int_1^\infty \mathrm{d}t\,\partial_t\, f(Yx^2t) = -\int_{x^2}^{\eta^2} \mathrm{d}t\partial_t'\, f(Yt') = -Y\partial_Y \int_{x^2}^{\eta^2} \frac{\mathrm{d}t'}{t'} f(Yt')$ and

$$\frac{\partial(I_2)}{\partial \not{\mathbf{p}}}\bigg|_{\not{\mathbf{p}}=m} = \frac{\imath}{8\pi^2}\lim_{\epsilon\to 0}\int_\epsilon^1 \mathrm{d}X(1-x)\int_0^\infty \mathrm{d}Y\partial_Y\left[\frac{Y^2(Y-1)}{(Y+1)^3}\right][\log\eta^2 - \log x^2],$$

the Y-integral is now convergent by itself and the PU-test functions have been extended to 1 over the whole domain of integration in Y with the result (cf footnote 3)

$$\frac{\partial(I_2)}{\partial \not{\mathbf{p}}}\,|_{\not{\mathbf{p}}=m} = \frac{\imath}{16\pi^2}[\log\eta^2 + 3]. \tag{D.2}$$

D.2. Contributions to the longitudinal term of the vacuum polarization, equations (3.8a) and (3.8b)

With the same technicalities as above, the first relevant contribution is

$$\frac{1}{2}\int \frac{\mathrm{d}^4\mathbf{p}}{(2\pi)^4}\frac{\mathbf{p}^2 f(\mathbf{p}^2)}{[\mathbf{p}^2 + m^2(x,\mathbf{q}^2)]^2} = -\frac{m^2(x,\mathbf{q}^2)}{2(4\pi)^2}\int_0^\infty \mathrm{d}Y\left[\frac{1}{(Y+1)} + \frac{Y}{(Y+1)^2}\right] f\left(Y\frac{m^2(x,\mathbf{q}^2)}{m^2}\right),$$

$$= -\frac{m^2(x,\mathbf{q}^2)}{2(4\pi)^2}\int_0^\infty \mathrm{d}Y\partial_Y\left[\frac{Y}{(Y+1)} + \frac{Y^2}{(Y+1)^2}\right] \times \int_{\frac{m^2(x,\mathbf{q}^2)}{m^2}}^{\eta^2} \frac{\mathrm{d}t}{t} f(Yt), \tag{D.3}$$

$$= -\frac{m^2(x,\mathbf{q}^2)}{(4\pi)^2}\log\left[\frac{\eta^2 m^2}{m^2(x,\mathbf{q}^2)}\right], \tag{D.4}$$

with $m^2(x,\mathbf{q}^2) = m^2 + \mathbf{q}^2 x(1-x) > 0$. The second contribution is

$$m^2(x,\mathbf{q}^2)\int \frac{\mathrm{d}^4\mathbf{p}}{(2\pi)^4}\frac{f(\mathbf{p}^2)}{[\mathbf{p}^2 + m^2(x,\mathbf{q}^2)]^2} = \frac{m^2(x,\mathbf{q}^2)}{(4\pi)^2}\int_0^\infty \mathrm{d}Y\left[\frac{Y}{(Y+1)^2}\right] f\left(Y\frac{m^2(x,\mathbf{q}^2)}{m^2}\right), \tag{D.5}$$

$$= \frac{m^2(x,\mathbf{q}^2)}{(4\pi)^2}\int_0^\infty \mathrm{d}Y\partial_Y\left[\frac{Y^2}{(Y+1)^2}\right]\int_{\frac{m^2(x,\mathbf{q}^2)}{m^2}}^{\eta^2}\frac{\mathrm{d}t}{t} f(Yt),$$

$$= \frac{m^2(x,\mathbf{q}^2)}{(4\pi)^2}\log\left[\frac{\eta^2 m^2}{m^2(x,\mathbf{q}^2)}\right], \tag{D.6}$$

which shows that the sum of (D.4) and (D.6) is zero, as stated in the main text.

D.3. QCD longitudinal contribution to the three-gluon and gluon–ghost vacuum polarization to one loop

We are concerned here with the determination of the function $J(\mathbf{q}^2, x)$ in equations (3.12*a*) and (3.12*b*):

$$J(\mathbf{q}^2, x) = \int \frac{\mathrm{d}^4\mathbf{p}}{(2\pi)^4}\frac{f(\mathbf{p}^2)}{[\mathbf{p}^2 + \mathbf{q}^2 x(1-x)]^2}. \tag{D.7}$$

At variance with equation (D.3) where a new integration variable Y could be used, for $m^2(x,\mathbf{q})$ is positive definite, the integral in (D.7) has a singularity at $\mathbf{p}^2 = 0$ when $x = (0, 1)$. Hence,

$x(1-x)$ cannot be collected in a new integration variable, only the external square-momentum $\mathbf{q}^2$ can. We have then, with $Xt = Z$

$$J(\mathbf{q}^2, x) = -\frac{1}{16\pi^2}\int_{\frac{\mathbf{q}^2}{\Lambda^2}}^{\eta^2}\frac{\mathrm{d}t}{t}\int_0^\infty \mathrm{d}X\frac{X}{[X+x(1-x)]^2}X\partial_X f(Xt),$$

$$= \frac{1}{16\pi^2}\int_{\frac{\mathbf{q}^2}{\Lambda^2}}^{\eta^2}\frac{\mathrm{d}t}{t}\int_0^\infty \mathrm{d}Z\partial_Z\left[\frac{Z^2}{(Z+tx(1-x))^2}\right]f(Z),$$

$$= \frac{1}{16\pi^2}\left(\log\eta^2 - \log\frac{\mathbf{q}^2}{\Lambda^2}\right),$$

which is independent of x and property (3.12*b*) follows.

References

[1] Grangé P and Werner E 2006 *Nucl. Phys.* B **161** 75

[2] Grangé P and Werner E 2011 *J. Phys. A: Math. Theor.* **44** 385402

[3] Grangé P, Mathiot J F, Mutet B and Werner E 2009 *Phys. Rev.* D **80** 105012

[4] Grangé P, Mathiot J F, Mutet B and Werner E 2010 *Phys. Rev.* D **82** 025012

[5] Grangé P and Werner E 2005 *Few-Body Syst.* **35** 103

[6] Fujikawa K 1979 *Phys. Rev. Lett.* **42** 1195
Fujikawa K 1980 *Phys. Rev.* D **21** 2848
Fujikawa K 1980 *Phys. Rev.* D **22** 1499

[7] 't Hooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189

[8] Capper D M, Jones D R T and Nieuwenhuizen P 1980 *Nucl. Phys.* **167** 479

[9] Piguet O and Sorella S P 1995 *Algebraic Renormalization, Perturbative Renormalization, Symmetries and Anomalies (Lecture Notes in Physics Monographs* vol 28*)* (Berlin: Springer) pp 1–133
Steinmann O 2000 *Perturbative Quantum Electrodynamics and Axiomatic Field Theory* (Berlin: Springer)
Grassi A, Hurt T and Steinhauser M 2001 *Ann. Phys.* **288** 197–248
Stora R 2008 *Int. J. Geom. Methods Mod. Phys.* **5** 1345

[10] Bogoliubov N N and Parasiuk O 1957 *Acta Math.* **97** 227
Hepp K 1966 *Commun. Math. Phys.* **2** 301
Hahn Y and Zimmermann W 1968 *Commun. Math. Phys.* **10** 330
Zimmermann W 1968 *Commun. Math. Phys.* **11** 1
Zimmermann W 1969 *Commun. Math. Phys.* **15** 208

[11] Schwartz L 1966 *Théorie des Distributions* (Paris: Hermann)

[12] Felsager B 1998 *Geometry, Particles, and Physics* (New York: Springer)

[13] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry (Interscience Tracts in Pure and Applied Mathematics* vol 15*)* (New York: Interscience)
Spivak M 1979 *A Comprehensive Introduction to Differential Geometry 2nd edn* (Boston, MA: Publish or Perish)
Nakahara M 1990 *Geometry, Topology and Physics (Graduate Student Series in Physics)* (Bristol: IOP Publishing)

[14] Bralić N E 1980 *Phys. Rev.* D **22** 3090

[15] Cartier P and DeWitt-Morette C 1995 *J. Math. Phys.* **36** 2236

[16] Vilkoviski V A 1984 *Quantum Theory of Gravity* ed S M Christensen (Bristol: Hilger)
DeWitt B and Molina C 1997 *Gauge Theory Without Ghosts in Functional Integration: Basics and Applications* (*NATO Advanced Study Institute Series*) ed P Cartier, C DeWitt-Morette and A Folacci (New York: Plenum)

[17] Bassetto A Nardelli G and Soldati R 1991 *Yang–Mills Theories in Algebraic Non-Covariant Gauges* (Singapore: World Scientific)

[18] Srivastava P P and Brodsky S 2001 *Phys. Rev.* D **64** 045006

[19] Das A Frenkel J and Perez S 2004 *Phys. Rev.* D **70** 125001
Das A and Frenkel J 2005 *Phys. Rev.* D **71** 087701

[20] Leibbrandt G 1994 *Quantization of Yang–Mills and Chern–Simons Theory in Axial-Type Gauges* (Singapore: World Scientific)

[21] Mutet B, Grangé P and Werner E 2008 Light-cone gauge singularities in the photon propagator and residual gauge transformations *Proc. Light Cone Relativistic Nuclear and Particle Physics* PoS(LC2008)005

[22] Estrada R and Kanwall K P 2002 *A Distributional Approach to Asymptotics: Theory and Applications* (Boston: Birkhaüser)
[23] Scharf G 1995 *Finite QED: The Causal Approach* (Berlin: Springer)
[24] Mutet B, Grangé P and Werner E 2008 Taylor–Lagrange renormalisation: electron and gauge-boson self-energies *Proc. Light Cone Relativistic Hadronic and Particle Physics* PoS(LC2010) 026
[25] Speer E R 1974 *J. Math. Phys.* **15** 1
[26] Breitenlohner P and Maison D 1977 *Commun. Math. Phys.* **52** 11
Breitenlohner P and Maison D 1977 *Commun. Math. Phys.* **52** 39
Breitenlohner P and Maison D 1977 *Commun. Math. Phys.* **52** 55
[27] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[28] Jackiw R 2000 *Int. J. Mod. Phys.* B **14** 2001
Hiller B, Mota A L, Nemes Osipov A A and Sampio M 2006 *Nucl. Phys.* A **769** 53
[29] Dütsch M and Fredenhagen K 2003 *Commun. Math. Phys.* **243** 275
Dütsch M and Fredenhagen K 2004 *Rev. Math. Phys.* **16** 1291
[30] Lowenstein J H 1971 *Phys. Rev.* D **4** 2281
Lowenstein J H, Weinstein M and Zimmermann W 1974 *Phys. Rev.* D **10** 1854
Lowenstein J H, Weinstein M and Zimmermann W 1974 *Phys. Rev.* D **10** 2500
[31] Epstein H and Glaser V 1973 *Ann. Inst. Henri Poincaré* A **19** 211
[32] Ryder L H 1988 *Quantum Field Theory* (Cambridge: Cambridge University Press)
[33] Collins J 1987 *Renormalization (Cambridge Monographs on Mathematical Physics)* (Cambridge: Cambridge University Press)
[34] Kanwall R P 1998 *Generalized Functions: Theory and Techniques* (Boston, MA: Birkhäuser)